DEVELOPMENTAL MATHEMATICS

SECOND EDITION

D. Franklin Wright

Instructor Preview

HAWKES LEARNING

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FOR REVIEW ONLY

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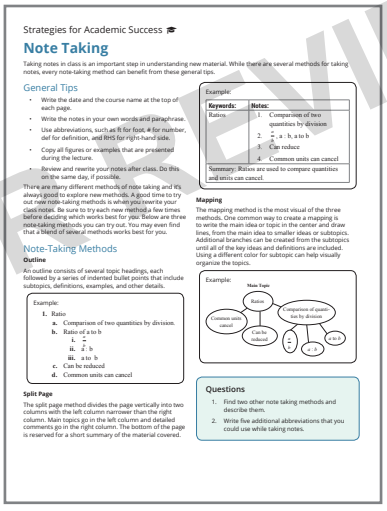
Developmental Mathematics: Content Highlights

New Features

Strategies for Academic Success

A new section has been included to help students hone their skills in note taking, time management, test taking, and reading. This section also provides tips for improving memory, overcoming test anxiety, and finding a math tutor. (See page 19 for more)



Strategies for Academic Success

Note Taking

Taking notes in class is an important step in understanding new material. While there are several methods for taking notes, every note-taking method can benefit from these general tips.

General Tips

- Write the date and the course name at the top of each page.
- Write the notes in your own words and paraphrase.
- Use abbreviations, such as # for foot, # for number, def for definition, and RHS for right-hand side.
- Copy all figures or examples that are presented during the lecture.
- Review and rewrite your notes after class. Do this on the same day, if possible.

There are many different methods of note taking and it's always good to explore new methods. A good time to try out new note-taking methods is when you rewrite your class notes. Be sure to try each new method a few times before deciding which works best for you. Below are three note-taking methods you can try out. You may even find that a blend of several methods works best for you.

Note-Taking Methods

Outline

An outline consists of several topic headings, each followed by a series of indented bullet points that include subtopics, definitions, examples, and other details.

Example:

1. Ratio
   a. Comparison of two quantities by division.
   b. Ratio of a to b
      i. $\frac{a}{b}$
      ii. $\frac{a}{b}$: b
      iii. a to b
   c. Can be reduced
   d. Common units can cancel

Split Page

The split page method divides the page vertically into two columns with the left column narrower than the right column. Main topics go in the left column and detailed comments go in the right column. The bottom of the page is reserved for a short summary of the material covered.

Example:

Keywords:	Notes:
Ratios	1. Comparison of two quantities by division 2. $\frac{a}{b}$: a : b, a to b 3. Can reduce 4. Common units can cancel
Summary: Ratios are used to compare quantities and units can cancel.	

Mapping

The mapping method is the most visual of the three methods. One common way to create a mapping is to write the main idea or topic in the center and draw lines, from the main idea to smaller ideas or subtopics. Additional branches can be created from the subtopics until all of the key ideas and definitions are included. Using a different color for subtopic can help visually organize the topics.

Example:

Questions

1. Find two other note taking methods and describe them.
2. Write five additional abbreviations that you could use while taking notes.

Chapter Projects

This new feature promotes collaboration and shows students the practical side of mathematics through activities using real-world applications of the concepts taught in the chapter. (See page 24 for more)

Chapter Project

Take Me Out to the Ball Game!

An activity to demonstrate the use of percents and percent of increase or decrease in real life.

The Atlanta Braves baseball team has been one of the most popular baseball teams for fans not only from Georgia, but throughout the Carolinas and the southeastern United States. The Braves franchise started playing at the Atlanta-Fulton County Stadium in 1966 and this continued to be their home field for 30 years. In 1996, the Centennial Olympic Stadium that was built for the 1996 Summer Olympics was converted to a new ballpark for the Atlanta Braves. The ballpark was named Turner Field and was opened for play in 1997. In 2017, the braves moved to a new stadium named SunTrust Park.

Round all percents to the nearest whole percent.

1. The Atlanta-Fulton County Stadium had a seating capacity of 52,769 fans. Turner Field had a seating capacity of 50,096. SunTrust Park has a seating capacity of 41,149.
   a. Determine the amount of decrease in seating capacity between Turner Field and the original Braves stadium
   b. Determine the percent decrease in seating capacity between SunTrust Park and Turner field.
2. The Centennial Olympic Stadium had approximately 85,000 seats. Some of the seating was removed in order to convert it to the Turner Field ballpark. Rounding the number of seats in Turner Field to the nearest thousand, what is the approximate percent decrease in seating capacity from the original Olympic stadium?
3. When Turner Field opened in 1997, the average attendance at an Atlanta Braves game was 42,771. In 2016 the average attendance was 24,950. What is the percent decrease in attendance from 1997 to 2016?
   Source: baseball-almanac.com
4. The highest average attendance for the Braves was 47,960 in 1993 at the Atlanta-Foulton County Stadium. The Lowest average attendance was 6642 in 1975 at the Atlanta-Foulton County Stadium. What is the percent increase from the lowest attendance to the highest?
   Source: baseball-almanac.com
5. Chipper Jones, a popular Braves third baseman, retired in July 2013. He started his career with the Braves in 1993 at the age of 21.
   Source: espn.go.com
   a. In 2001, Chipper had 189 hits in 572 at-bats. Calculate Chipper's batting average for the season by dividing the number of hits by the number of at-bats. Round to 3 decimal places.
   b. In 2008, Chipper had 160 hits in 439 at-bats. Calculate Chipper's batting average for the season by dividing the number of hits by the number of at-bats. Round to 3 decimal places.
   c. Calculate the percent change in Chipper's batting average from 2001 to 2008.
   d. Does this represent a percent increase or decrease?
6. In 2001, Chipper had 102 RBIs (runs batted in). In 2008, Chipper had only 75 RBIs.
   a. Calculate the percent change in RBIs from 2001 to 2008.
   b. Does this represent a percent increase or decrease?

Concept Check

New exercises to assess students' conceptual understanding of topics and important definitions are included in every section.

> **1.** Every square is a rectangle but not every rectangle is a _____.

Applications

Additional real-world application problems have been added throughout the text to challenge students to apply the concepts taught in the lesson.

Extra Material

Additional, more advanced topics have been added to provide students with a text that fully prepares them for future college mathematics courses.

Additional Features

Math at Work

Each chapter begins with a brief discussion related to a concept developed in the coming material and includes questions students will solve later in the chapter to solidify their knowledge and understanding.

Objectives

The objectives provide students with a clear and concise list of the main concepts and methods taught in each section, enabling students to focus their time and effort on the most important topics. Objectives have corresponding labels located in the section text where the topic is introduced for ease of reference.

Objectives

A. Multiply fractions.

B. Reduce fractions to lowest terms.

C. Multiply and reduce fractions to lowest terms.

A First objective

Examples

Examples are denoted with titled headers indicating the problem-solving skill being presented. Each section contains carefully explained examples with appropriate tables, diagrams, and graphs. Examples are presented in an easy-to-understand, step-by-step fashion and annotated with notes for additional clarification.

Example 1 Multiplying Fractions

Multiply: $\frac{6}{7} \cdot \frac{8}{5}$

Solution

$$\frac{6}{7} \cdot \frac{8}{5} = \frac{6 \cdot 8}{7 \cdot 5} = \frac{48}{35}$$

Now work margin exercise 1.

Margin Exercises

Each example has a corresponding margin exercise to test students' understanding of what was taught in the example.

1. Solve: $3x + 4 = 7$

Answers

1. $x = 1$

Notes

Notes boxes in the margin point out important information that will help deepen student understanding of the topics. Often these are helpful hints about subtle details in the definitions that many students do not notice upon first glance.

Notes

Greek mathematician Euclid is often referred to as the 'Father of Geometry' for his revolutionary ideas and influential textbook called *Elements* that he wrote around the year 300 BC.

Definition Boxes

Straightforward definitions are presented in highly visible boxes for easy reference.

Algebra

The branch of mathematics that deals with general statements of relations, utilizing letters and other symbols to represent specific sets of numbers, values, vectors, and so on, in the description of such relations.

DEFINITION

Common Errors

These hard-to-miss boxes highlight common mistakes and how to avoid them.

Attention

Don't forget to carry the 1!

Calculators

For visual learners, key strokes and screenshots are provided when appropriate for visual reference. We also provide step-by-step instructions for using a simple four-function calculator for more basic operations, as well as a TI-84 Plus for graphing skills.

CALCULATORS

Performing a task with a calculator

Press the keys 6 7 9 × 2 + 3 8 0.
Then press =.

The display will read 1738.

Exercises

Each section includes a variety of exercises to give the students much-needed practice applying and reinforcing the skills they learned in the section. The exercises progress from relatively easy problems to more difficult problems.

Writing and Thinking

This feature gives students an opportunity to independently explore and expand on concepts presented in the chapter. These questions foster a better understanding of the concepts learned within each section.

Collaborative Learning

This feature encourages students to work with others to further explore and apply concepts learned in the chapter. These questions help students realize that they see many mathematical concepts in the world around them every day.

Index of Key Ideas and Terms

Each chapter contains an index highlighting the main concepts within the chapter. This summary gives complete definitions and concise steps to solve particular types of problems.

Chapter Tests

Each chapter includes a chapter test that provides an opportunity for the students to practice the skills presented in the chapter in a test format.

Answer Key

Located in the back of the book, the answer key provides answers to all odd numbered exercises in each section, as well as all answers to the exercises in the Chapter Tests. This allows students to check their work to ensure that they are accurately applying the methods and skills they have learned.

Strategies for Academic Success

How to Read a Math Textbook

Reading a textbook is very different than reading a book for fun. You have to concentrate more on what you are reading because you will likely be tested on the content. Reading a math textbook requires a different approach than reading literature or history textbooks because the math textbook contains a lot of symbols and formulas in addition to words. Here are some tips to help you successfully read a math textbook.

Don't Skim

When reading math textbooks, look at everything: titles, learning objectives, definitions, formulas, text in the margins, and any text that is highlighted, outlined, or in bold. Also pay close attention to any tables, figures, charts, and graphs.

Minimize Distractions

Reading a math textbook requires much more concentration than a novel by your favorite author, so pick a study environment with few distractions and a time when you are most attentive.

Start at the Beginning

Don't start in the middle of an assigned section. Math tends to build on previously learned concepts and you may miss an important concept or formula that is crucial to understanding the rest of the material in the section.

Highlight and Annotate

Put your book to good use and don't be afraid to add comments and highlighting. If you don't understand something in the text, reread it a couple of times. If it is still not clear, note the text with a question mark or some other notation so you can ask your instructor about it.

Go through Each Step of Each Example

Make sure you understand each step of an example. If you don't understand something, mark it so you can ask about it in class. Sometimes math textbooks leave out intermediate steps to save space. Try working through the examples on your own, filling in any missing steps.

Take Notes *< This is important!*

Write down important definitions, symbols or notation, properties, formulas, theorems, and procedures. Review these daily as you do your homework and before taking quizzes and tests. Practice rewriting definitions in your own words so you understand them better.

Notes 9-25-17:

- The opposite of a negative integer is a positive integer.

- To add tow integers with the same signs add their absolute values and use their common sign

Use Available Resources

Many textbooks have companion websites to help you understand the content. These resources may contain videos that help explain more complex steps or concepts. Try searching the internet for additional explanations of topics you don't understand.

Read the Material Before Class

Try to read the material from your book before the instructor lectures on it. After the lecture, reread the section again to help you retain the information as you look over your class notes.

Understand the Mathematical Definitions + × =

Many terms used in everyday English have a different meaning when used in mathematics. Some examples include equivalent, similar, average, median, and product. Two equations can be equivalent to one another without being equal. An average can be computed mathematically in several ways. It is important to note these differences in meaning in your notebook along with important definitions and formulas.

Try Reading the Material Aloud

Reading aloud makes you focus on every word in the sentence. Leaving out a word in a sentence or math problem could give it a totally different meaning, so be sure to read the text carefully and reread, if necessary.

Questions

1. Explain how taking notes can help you understand new concepts and skills while reading a math textbook.
2. Think of two more tips for reading a math textbook.

Strategies for Academic Success

Tips for Success in a Math Course

Read Your Textbook/Workbook

One of the most important skills when taking a math class is knowing how to read a math textbook. Reading a section before class and then reading it again afterwards is an important strategy for success in a math course. If you don't have time to read the entire assigned section, you can get an overview by reading the introduction or summary and looking at section objectives, headings, and vocabulary terms.

Take Notes

Take notes in class using a method that works for you. There are many different note-taking strategies, such as the Cornell Method and Concept Mapping. You can try researching these and other methods to see if they might work better than your current note-taking system.

Review

While the information is fresh in your mind, read through your notes as soon as possible after class to make sure they are readable, write down any questions you have, and fill in any gaps. Mark any information that is incomplete so that you can get it from the textbook or your instructor later.

Stay Organized

As you review your notes each day, be sure to label them using categories such as definition, theorem, formula, example, and procedure. Try highlighting each category with a different colored highlighter.

Use Study Aids

Use note cards to help you remember definitions, theorems, formulas, or procedures. Use the front of the card for the vocabulary term, theorem name, formula name, or procedure description. Write the definition, the theorem, the formula, or the procedure on the back of the card, along with a description in your own words.

Practice, Practice, Practice!

Math is like playing a sport. You can't improve your basketball skills if you don't practice—the same is true of math. Math can't be learned by only watching your instructor work through problems; you have to be actively involved in doing the math yourself. Work through the examples in the book, do some practice exercises at the end of the section or chapter, and keep up with homework assignments on a daily basis.

Do Your Homework

When doing homework, always allow plenty of time to finish it before it is due. Check your answers when possible to make sure they are correct. With word or application problems, always review your answer to see if it appears reasonable. Use the estimation techniques that you have learned to determine if your answer makes sense.

Understand, Don't Memorize

Don't try to memorize formulas or theorems without understanding them. Try describing or explaining them in your own words or look for patterns in formulas so you don't have to memorize them. For example, you don't need to memorize every perimeter formula if you understand that perimeter is equal to the sum of the lengths of the sides of the figure.

Study

Plan to study two to three hours outside of class for every hour spent in class. If math is your most difficult subject, then study while you are alert and fresh. Pick a study time when you will have the least interruptions or distractions so that you can concentrate.

Manage Your Time

Don't spend more than 10 to 15 minutes working on a single problem. If you can't figure out the answer, put it aside and work on another one. You may learn something from the next problem that will help you with the one you couldn't do. Mark the problems that you skip so that you can ask your instructor about it during the next class. It may also help to work a similar, but perhaps easier, problem.

Questions

1. Based on your schedule, what are the best times and places for you to study for this class?
2. Describe your method of taking notes. List two ways to improve your method.

Strategies for Academic Success

Tips for Improving Math Test Scores

Preparing for a Math Test

- Avoid cramming right before the test and don't wait until the night before to study. Review your notes and note cards every day in preparation for quizzes and tests.
- If the textbook has a chapter review or practice test after each chapter, work through the problems as practice for the test.
- If the textbook has accompanying software with review problems or practice tests, use it for review.
- Review and rework homework problems, especially the ones that you found difficult.
- If you are having trouble understanding certain concepts or solving any types of problems, schedule a meeting with your instructor or arrange for a tutoring session (if your college offers a tutoring service) well in advance of the next test.

Test-Taking Strategies

- Scan the test as soon as you get it to determine the number of questions, their levels of difficulty, and their point values so you can adequately gauge how much time you will have to spend on each question.
- Start with the questions that seem easiest or that you know how to work immediately. If there are problems with large point values, work them next since they count for a larger portion of your grade.
- Show all steps in your math work. This will make it quicker to check your answers later once you are finished since you will not have to work through all the steps again.
- If you are having difficulty remembering how to work a problem, skip it and come back to it later so that you don't spend all of your time on one problem.

After the Test

- The material learned in most math courses is cumulative, which means any concepts you miss on each test may be needed to understand concepts in future chapters. That's why it is extremely important to review your returned tests and correct any misunderstandings that may hinder your performance on future tests.
- Be sure to correct any work you did wrong on the test so that you know the correct way to do the problem in the future. If you are not sure what you did wrong, get help from a peer who scored well on the test or schedule time with your instructor to go over the test.
- Analyze the test questions to determine if the majority came from your class notes, homework problems, or the textbook. This will give you a better idea of how to spend your time studying for the next test.
- Analyze the errors you made on the test. Were they careless mistakes? Did you run out of time? Did you not understand the material well enough? Were you unsure of which method to use?
- Based on your analysis, determine what you should do differently before the next test and where you should focus your time.

Questions

1. Determine the resources that are available to you to help you prepare for tests, such as instructor office hours, tutoring center hours, and study groups.
2. Discuss two additional test taking strategies.

Strategies for Academic Success

Practice, Patience, and Persistence!

Have you ever heard the phrase "practice makes perfect"? This saying applies to many things in life. You won't become a concert pianist without many hours of practice. You won't become an NBA basketball star by sitting around and watching basketball on TV. The saying even applies to riding a bike. You can watch all of the videos and read all of the books on riding a bike, but you won't learn how to ride a bike without actually getting on the bike and trying to do it yourself. The same idea applies to math. Math is not a spectator sport.

Math is not learned by sleeping with your math book under your pillow at night and hoping for osmosis (a scientific term implying that math knowledge would move from a place of higher concentration—the math book—to a place of lower concentration—your brain). You also don't learn math by watching your professor do hundreds of math problems while you sit and watch. Math is learned by doing. Not just by doing one or two problems, but by doing many problems. Math is just like a sport in this sense. You become good at it by doing it, not by watching others do it. You can also think of learning math like learning to dance. A famous ballerina doesn't take a dance class or two and then end up dancing the lead in The Nutcracker. It takes years of practice, patience, and persistence to get that part.

Now, we aren't suggesting that you dedicate your life to doing math, but at this point in your education, you've already spent quite a few years studying the subject. You will continue to do math throughout college—and your life. To be able to financially support yourself and your family, you will have to find a job, earn a salary, and invest your money—all of which require some ability to do math. You may not think so right now, but math is one of the more useful subjects you will study.

It's important not only to practice math when taking a math course, but also to be patient and not expect immediate success. Just like a ballerina or NBA basketball star, who didn't become exceptional athletes overnight, it will take some time and patience to develop your math skills. Sure, you will make some mistakes along the way, but learn from those mistakes and move on.

Practice, patience, and persistence are especially important when working through applications or word problems. Most students don't like word problems and, therefore, avoid them. You won't become good at working word problems unless you practice them over and over again. You'll need to be patient when working through word problems in math since they will require more time to work than typical math skills exercises. The process of solving word problems is not a quick one and will take patience and persistence on your part to be successful.

Just as you work your body through physical exercise, you have to work your brain through mental exercise. Math is an excellent subject to provide the mental exercise needed to stimulate your brain. Your brain is flexible and it continues to grow throughout your life span—but only if provided the right stimuli. Studying mathematics and persistently working through tough math problems is one way to promote increased brain function. So, when doing mathematics, remember the 3 P's—Practice, Patience, and Persistence—and the positive effects they will have on your brain!

Questions

1. What is another area (not mentioned here) that requires practice, patience, and persistence to master? Can you think of anything you could master without practice?
2. Can you think of an example in your study of math where practice, patience, and persistence have helped you improve?

Strategies for Academic Success

Note Taking

Taking notes in class is an important step in understanding new material. While there are several methods for taking notes, every note-taking method can benefit from these general tips.

General Tips

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- Write the notes in your own words and paraphrase.
- Use abbreviations, such as ft for foot, # for number, def for definition, and RHS for right-hand side.
- Copy all figures or examples that are presented during the lecture.
- Review and rewrite your notes after class. Do this on the same day, if possible.

There are many different methods of note taking and it's always good to explore new methods. A good time to try out new note-taking methods is when you rewrite your class notes. Be sure to try each new method a few times before deciding which works best for you. Presented here are three note-taking methods you can try out. You may even find that a blend of several methods works best for you.

Note-Taking Methods

Outline

An outline consists of several topic headings, each followed by a series of indented bullet points that include subtopics, definitions, examples, and other details.

Example:

1. Ratio
   - **a.** Comparison of two quantities by division.
   - **b.** Ratio of a to b
     - **i.** $\frac{a}{b}$
     - **ii.** $a : b$
     - **iii.** a to b
   - **c.** Can be reduced
   - **d.** Common units can cancel

Split Page

The split page method divides the page vertically into two columns with the left column narrower than the right column. Main topics go in the left column and detailed comments go in the right column. The bottom of the page is reserved for a short summary of the material covered.

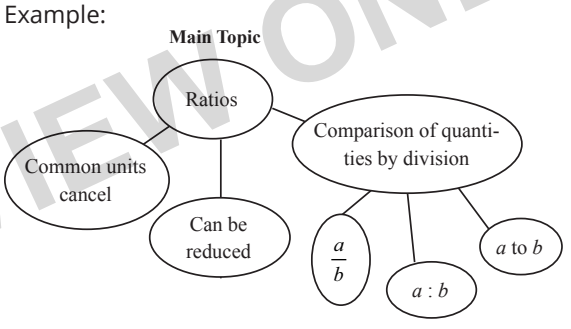
Example:

Keywords:	**Notes:**
Ratios	1. Comparison of two quantities by division 2. $\frac{a}{b}$, $a : b$, a to b 3. Can reduce 4. Common units can cancel
Summary: Ratios are used to compare quantities and units can cancel.	

Mapping

The mapping method is the most visual of the three methods. One common way to create a mapping is to write the main idea or topic in the center and draw lines, from the main idea to smaller ideas or subtopics. Additional branches can be created from the subtopics until all of the key ideas and definitions are included. Using a different color for subtopic can help visually organize the topics.

Example:



Questions

1. Find two other note taking methods and describe them.
2. Write five additional abbreviations that you could use while taking notes.

Strategies for Academic Success

Do I Need a Math Tutor?

If you do not understand the material being presented in class, if you are struggling with completing homework assignments, or if you are doing poorly on tests, then you may need to consider getting a tutor. In college, everyone needs help at some point in time. What's important is to recognize that you need help before it's too late and you end up having to retake the class.

Alternatives to Tutoring

Before getting a tutor, you might consider setting up a meeting with your instructor during their office hours to get help. Unfortunately, you may find that your instructor's office hours don't coincide with your schedule or don't provide enough time for one-on-one help.

Another alternative is to put together a study group of classmates from your math class. Working in groups and explaining your work to others can be very beneficial to your understanding of mathematics. Study groups work best if there are three to six members. Having too many people in a study group may make it difficult to schedule a time for all group members to meet. A large study group may also increase distractions. If you have too few people and those that attend are just as lost as you, then you aren't going to be helpful to each other.

Where to Find a Tutor

Many schools have both group and individual tutoring available. In most cases, the cost of this tutoring is included in tuition costs. If your college offers tutoring through a learning lab or tutoring center, then you should take advantage of it. You may need to complete an application to be considered for tutoring, so be sure to get the necessary paperwork at the start of each semester to increase your chances of getting a tutoring time that works well with your schedule. This is especially important if you know that you struggle with math or haven't taken any math classes in a while.

If you find that you need more help than the tutoring center can provide, or your school doesn't offer tutoring, you can hire a private tutor. The hourly cost to hire a private tutor varies significantly depending on the area you live in along with the education and experience level of the tutor. You might be able to find a tutor by asking your instructor for references or by asking friends who have taken higher-level math classes than you have. You can also try researching the internet for local reputable tutoring organizations in your area.

What to Look for in a Tutor

Whether you obtain a tutor through your college or hire a personal tutor, look for someone who has experience, educational qualifications, and who is friendly and easy to work with. If you find that the tutor's personality or learning style isn't similar to yours, then you should look for a different tutor that matches your style. It may take some effort to find a tutor who works well with you.

How to Prepare for a Tutoring Session

To get the most out of your tutoring session, come prepared by bringing your text, class notes, and any homework or questions you need help with. If you know ahead of time what you will be working on, communicate this to the tutor so they can also come prepared. You should attempt the homework prior to the session and write notes or questions for the tutor. Do not use the tutor to do your homework for you. The tutor will explain to you how to do the work and let you work some problems on your own while he or she observes. Ask the tutor to explain the steps aloud while working through a problem. Be sure to do the same so that the tutor can correct any mistakes in your reasoning. Take notes during your tutoring session and ask the tutor if he or she has any additional resources such as websites, videos, or handouts that may help you.

Questions

1. It's important to find a tutor whose learning style is similar to yours. What are some ways that learning styles can be different?
2. What sort of tutoring services does your school offer?

Strategies for Academic Success

Tips for Improving Your Memory

Experts believe that there are three ways that we store memories: first in the sensory stage, then in short term memory, and finally in long term memory.[1] Because we can't retain all the information that bombards us daily, the different stages of memory act as a filter. Your sensory memory lasts only a fraction of a second and holds your perception of a visual image, a sound, or a touch. The sensation then moves to your short term memory, which has the limited capacity to hold about seven items for no more than 20 to 30 seconds at a time. Important information is gradually transferred to long term memory. The more the information is repeated or used, the greater the chance that it will end up in long term memory. Unlike sensory and short term memory, long term memory can store unlimited amounts of information indefinitely. Here are some tips to improve your chances of moving important information to long-term memory.

Be attentive and focused on the information.

Study in a location that is free of distractions and avoid watching TV or listening to music with lyrics while studying.

Recite information aloud.

Ask yourself questions about the material to see if you can recall important facts and details. Pretend you are teaching or explaining the material to someone else. This will help you put the information into your own words.

Associate the information with something you already know.

Think about how you can make the information personally meaningful—how does it relate to your life, your experiences, and your current knowledge? If you can link new information to memories already stored, you create "mental hooks" that help you recall the information. For example, when trying to remember the formula for slope using rise and run, remember that rise would come alphabetically before run, so rise will be in the numerator in the slope fraction and run will be in the denominator.

Use visual images like diagrams, charts, and pictures.

You can make your own pictures and diagrams to help you recall important definitions, theorems, or concepts.

Split larger pieces of information into smaller "chunks."

This is useful when remembering strings of numbers, such as social security numbers and telephone numbers. Instead of remembering a sequence of digits such as 555777213 you can break it into chunks such as 555 777 213.

Group long lists of information into categories that make sense.

For example, instead of remembering all the properties of real numbers individually, try grouping them into shorter lists by operation, such as addition and multiplication.

Use mnemonics or memory techniques to help remember important concepts and facts.

A mnemonic that is commonly used to remember the order of operations is "Please Excuse My Dear Aunt Sally," which uses the first letter of the words Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction to help you remember the correct order to perform basic arithmetic calculations. To make the mnemonic more personal and possibly more memorable, make up one of your own.

Use acronyms to help remember important concepts or procedures.

An acronym is a type of mnemonic device which is a word made up by taking the first letter from each word that you want to remember and making a new word from the letters. For example, the word HOMES is often used to remember the five Great Lakes in North America where each letter in the word represents the first letter of one of the lakes: Huron, Ontario, Michigan, Erie, and Superior.

Questions

1. Create an original mnemonic or acronym for any math topic covered so far in this course.
2. Explain two ways you can incorporate these tips into your study routine.

1 Source: http://science.howstuffworks.com/life/inside-the-mind/human-brain/human-memory2.htm

Strategies for Academic Success

Overcoming Anxiety

People who are anxious about math are often just not good at taking math tests. If you understand the math you are learning but don't do well on math tests, you may be in the same situation. If there are other subject areas in which you also perform poorly on tests, then you may be experiencing test anxiety.

How to Reduce Math Anxiety

- Learn effective math study skills. Sit near the front of your class and take notes. Ask questions when you don't understand the material. Review your notes after class and read new material before it's covered in class. Keep up with your assignments and do a lot of practice problems.
- Don't accept negative self talk such as "I am not good at math" or "I just don't get it and never will." Maintain a positive attitude and set small math achievement goals to keep you positively moving toward bigger goals.
- Visualize yourself doing well in math, whether it's on a quiz or test, or passing a math class. Rehearse how you will feel and perform on an upcoming math test. It may also help to visualize how you will celebrate your success after doing well on the test.
- Form a math study group. Working with others may help you feel more relaxed about math in general and you may find that other people have the same fears.
- If you panic or freeze during a math test, try to work around the panic by finding something on the math test that you can do. Once you gain confidence, work through other problems you know how to do. Then, try completing the harder problems, knowing that you have a large part of the test completed already.
- If you have trouble remembering important concepts during tests, do what is called a "brain drain" and write down all the formulas and important facts that you have studied on your test or scratch paper as soon as you are given the test. Do this before you look at any questions on the test. Having this information available to you should help boost your confidence and reduce your anxiety. Doing practice brain drains while studying can help you remember the concepts when the test time comes.

How to Reduce Test Anxiety

- Be prepared. Knowing you have prepared well will make you more confident and less anxious.
- Get plenty of sleep the night before a big test and be sure to eat nutritious meals on the day of the test. It's helpful to exercise regularly and establish a set routine for test days. For example, your routine might include eating your favorite food, putting on your lucky shirt, and packing a special treat for after the test.
- Talk to your instructor about your anxiety. Your instructor may be able to make accommodations for you when taking tests that may make you feel more relaxed, such as extra time or a more calming testing place.
- Learn how to manage your anxiety by taking deep, slow breaths and thinking about places or people who make you happy and peaceful.
- When you receive a low score on a test, take time to analyze the reasons why you performed poorly. Did you prepare enough? Did you study the right material? Did you get enough rest the night before? Resolve to change those things that may have negatively affected your performance in the past before the next test.
- Learn effective test taking strategies. See the study skill on Tips for Improving Math Test Scores.

Questions

1. Describe your routine for test days. Think of two ways you can improve your routine to reduce stress and anxiety.
2. Research and describe the accommodations that your instructor or school can provide for test taking.

Strategies for Academic Success

Online Resources

With the invention of the internet, there are numerous resources available to students who need help with mathematics. Here are some quality online resources that we recommend.

HawkesTV

tv.hawkeslearning.com

If you are looking for instructional videos on a particular topic, then start with HawkesTV. There are hundreds of videos that can be found by looking under a particular math subject area such as introductory algebra, precalculus, or statistics. You can also find videos on study skills.

YouTube

www.youtube.com

You can also find math instructional videos on YouTube, but you have to search for videos by topic or key words. You may have to use various combinations of key words to find the particular topic you are looking for. Keep in mind that the quality of the videos varies considerably depending on who produces them.

Google Hangouts

plus.google.com/hangouts

You can organize a virtual study group of up to 10 people using Google Hangouts. This is a terrific tool when schedules are hectic and it avoids everyone having to travel to a central location. You do have to set up a Google+ profile to use Hangouts. In addition to video chat, the group members can share documents using Google Docs. This is a great tool for group projects!

Wolfram|Alpha

www.wolframalpha.com

Wolfram|Alpha is a computational knowledge engine developed by Wolfram Research that answers questions posed to it by computing the answer from "curated data." Typical search engines search all of the data on the Internet based on the key words given and then provide a list of documents or web pages that might contain relevant information. The data used by Wolfram|Alpha is said to be "curated" because someone has to verify its integrity before it can be added to the database, therefore ensuring that the data is of high quality. Users can submit questions and request calculations or graphs by typing their request into a text field. Wolfram|Alpha then computes the answers and related graphics from data gathered from both academic and commercial websites such as the CIA's World Factbook, the United States Geological Survey, financial data from Dow Jones, etc. Wolfram|Alpha uses the basic features of Mathematica, which is a computational toolkit designed earlier by Wolfram Research that includes computer algebra, symbol and number computation, graphics, and statistical capabilities.

Questions

1. Describe a situation where you think Wolfram|Alpha might be more helpful than YouTube, and vice versa.
2. What are some pros and cons to using Google Hangouts?

Strategies for Academic Success

Preparing for a Final Math Exam

Since math concepts build on one another, a final exam in math is not one you can study for in a night or even a day or two. To pull all the concepts together for the semester, you should plan to start one or two weeks ahead of time. Being comfortable with the material is key to going into the exam with confidence and lowering your anxiety.

Before You Start Preparing for the Exam

1. What is the date, time, and location of the exam? Check your syllabus for the final exam time and location. If it's not on your syllabus, your instructor should announce this information in class.
2. Is there a time limit on the exam? If you experience test anxiety on timed tests, be sure to speak to your professor about it and see if you can receive accommodations that will help reduce your anxiety, such as extended time or an alternate testing location.
3. Will you be able to use a formula sheet, calculator, and/or scrap paper on the exam? If you are not allowed to use a formula sheet, you should write down important formulas and memorize them. Most of the time, math professors will advise you of the formulas you need to know for an exam. If you cannot use a calculator on the exam, be sure to practice doing calculations by hand when you are preparing for the exam and go back and check them using the calculator.

A Week Before the Exam

1. Decide where to study for the exam and with whom. Make sure it's a comfortable study environment with few outside distractions. If you are studying with others, make sure the group is small and that the people in the group are motivated to study and do well on the exam. Plan to have snacks and water with you for energy and to avoid having to delay studying to go get something to eat or drink. Be sure and take small breaks every hour or two to keep focused and minimize frustration.
2. Organize your class notes and any flash cards with vocabulary, formulas, and theorems. If you haven't used flash cards for vocabulary, go back through your notes and highlight the vocabulary. Create a formula sheet to use on the exam, if the professor allows. If not, then you can use the formula sheet to memorize the formulas that will be on the exam.
3. Start studying for the exam. Studying a week before the exam gives you time to ask your instructor questions as you go over the material. Don't spend a lot of time reviewing material you already know. Go over the most difficult material or material that you don't understand so you can ask questions about it. Be sure to review old exams and work through any questions you missed.

3 Days Before the Exam

1. Make yourself a practice test consisting of the problem types. Don't necessarily put the questions in the order that the professor covered them in class.
2. Ask your instructor or classmates any questions that you have about the practice test so that you have time to go back and review the material you are having difficulty with.

The Night Before the Exam

1. Make sure you have all the supplies you will need to take the exam: formula sheet and calculator, if allowed, scratch paper, plain and colored pencils, highlighter, erasers, graph paper, extra batteries, etc.
2. If you won't be allowed to use your formula sheet, review it to make sure you know all the formulas. Right before going to bed, review your notes and study materials, but do not stay up all night to "cram."
3. Go to bed early and get a good night's sleep. You will do better if you are rested and alert.

The Day of the Exam

1. Get up with plenty of time to get to your exam without rushing. Eat a good breakfast and don't drink too much caffeine, which can make you anxious.
2. Review your notes, flash cards, and formula sheet again, if you have time.
3. Get to class early so you can be organized and mentally prepared.

Checklist for the Exam

Date of the Exam: _____ **Time of the Exam:** _____

Location of the Exam: _____

Items to bring to the exam:

___ calculator and extra batteries

___ formula sheet

___ scratch paper

___ graph paper

___ pencils

___ eraser

___ colored pencils or highlighter

___ ruler or straightedge

Notes or other things to remember for exam day:

During the Exam

1. Put your name at the top of your exam immediately. If you are not allowed to use a formula sheet, before you even look at the exam, do what is called a "brain drain" or "data dump." Recall as much of the information on your formula sheet as you possibly can and write it either on the scratch paper or in the exam margins if scratch paper is not allowed. You have now transferred over everything on your "mental cheat sheet" to the exam to help yourself as you work through the exam.
2. Read the directions carefully as you go through the exam and make sure you have answered the questions being asked. Also, check your solutions as you go. If you do any work on scratch paper, write down the number of the problem on the paper and highlight or circle your answer. This will save you time when you review the exam. The instructor may also give you partial credit for showing your work. (Don't forget to attach your scratch work to your exam when you turn it in.)
3. Skim the questions on the exam, marking the ones you know how to do immediately. These are the problems you will do first. Also note any questions that have a higher point value. You should try to work these next or be sure to leave yourself plenty of time to do them later.
4. If you get to a problem you don't know how to do, skip it and come back after you finish all the ones you know how to do. A problem you do later may jog your memory on how to do the problem you skipped.
5. For multiple choice questions, be sure to work the problem first before looking at the answer choices. If your answer is not one of the choices, then review your math work. You can also try starting with the answer choices and working backwards to see if any of them work in the problem. If this doesn't work, see if you can eliminate any of the answer choices and make an educated guess from the remaining ones. Mark the problem to come back to later when you review the exam.
6. Once you have an answer for all the problems, review the entire exam. Try working the problems differently and comparing the results or substituting the answers into the equation to verify they are correct. Do not worry about finishing early. You are in control of your own time—and your own success!

Questions

1. Does your syllabus provide any of the information needed for the checklist?
2. Are there any tips or suggestions mentioned here that you haven't thought of before?

Chapter Project

Before and After

An activity to demonstrate the use of geometric concepts in real life.

Suppose HGTV came to your home one day and said, "Congratulations, you have just won a FREE makeover for any room in your home! The only catch is that you have to determine the amount of materials needed to do the renovations and keep the budget under $2000." Could you pass up a deal like that? Would you be able to calculate the amount of flooring and paint needed to remodel the room? Remember it's a FREE makeover if you can!

Let's take an average size room that is rectangular in shape and measures 16 feet 3 inches in width by 18 feet 9 inches in length. The height of the ceiling is 8 feet. The plan is to repaint all the walls and the ceiling and to replace the carpet on the floor with hardwood flooring. You are also going to put crown molding around the top of the walls for a more sophisticated look.

1. Take the length and width measurements that are in feet and inches and convert them to a fractional number of feet and reduce to lowest terms. (Remember that there are 12 inches in a foot.)
2. Now convert these same measurements to decimal numbers.
3. Determine the number of square feet of flooring needed to redo the floor. (Express your answer in terms of a decimal and do not round the number.)
4. If the flooring comes in boxes that contain 24 square feet, how many boxes of flooring will be needed? (Remember that the store only sells whole boxes of flooring.)
5. If the flooring you have chosen costs $74.50 per box, how much will the hardwood flooring for the room cost (before sales tax)?
6. Figure out the surface area of the four walls and the ceiling that need to be painted, based on the room's dimensions. (We will ignore any windows, doors, or closets since this is an estimate.)
7. Assume that a gallon of paint covers 350 square feet and you are going to have to paint the walls and the ceilings twice to cover the current paint color. Determine how many gallons of paint you need to paint the room. (Again, assume that you can only buy whole gallons of paint. Any leftover paint can be used for touch-ups.)
8. If the paint you have chosen costs $18.95 per gallon, calculate the cost of the paint (before sales tax).
9. Determine how many feet of crown molding will be needed to go around the top of the room.
10. The molding comes in 12-foot sections only. How many sections will you need to buy?
11. If the molding costs $2.49 per linear foot, determine the cost of the molding (before sales tax).
12. Calculate the cost of all the materials for the room makeover (before sales tax).
    a. Were you able to stay within budget for the project?
    b. If so, then what extras could you add? If not, what could you adjust in this renovation to stay within budget?
    c. Using sales tax in your area, calculate the final price of the room makeover with sales tax included.

2.1 Introduction to Fractions and Mixed Numbers

Objectives

A. Understand the basic concepts of fractions.

B. Graph fractions on a number line.

C. Understand the basic concepts of mixed numbers.

D. Graph mixed numbers on a number line.

E. Change mixed numbers to improper fractions.

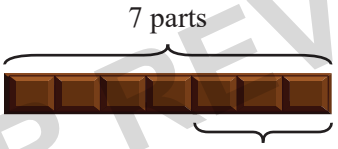
F. Change improper fractions to mixed numbers.

A Introduction to Fractions

Numbers such as $\frac{2}{3}$ (read "two-thirds") are said to be in **fraction form**. The top number, 2, is called the **numerator** and the bottom number, 3, is called the **denominator**.

$$\frac{2}{3} \quad \begin{matrix} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{matrix}$$

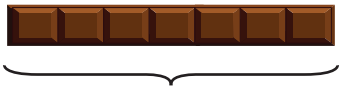
Fractions can be used to indicate parts of a whole. For example, if a whole candy bar has 7 equal parts, then the fraction $\frac{3}{7}$ (read "three-sevenths") indicates that we are considering 3 of those parts.



$\frac{3}{7}$ ← indicates 3 of 7 equal parts

Figure 1

The whole candy bar can be represented as $\frac{7}{7}$.

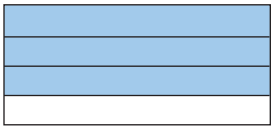


$\frac{7}{7} = 1$

Figure 2

Example 1 shows several fractions indicating parts of a whole.

Example 1 Understanding Fractions



Write a fraction indicating

a. the shaded part of the rectangle and

b. the unshaded part of the rectangle.

Solution

a. In the rectangle, 3 of the 4 equal parts are shaded. Thus, $\frac{3}{4}$ of the rectangle is shaded.

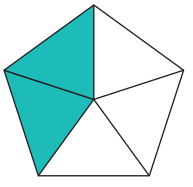
b. $\frac{1}{4}$ is not shaded.

Now work margin exercise 1.

1. Write a fraction indicating

a. the shaded part of the figure and

b. the unshaded part of the figure.



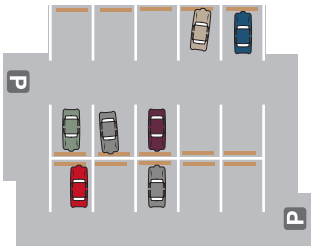
Answers

1. a. $\frac{2}{5}$ **b.** $\frac{3}{5}$

2. Write a fraction indicating

a. the portion of the parking spaces that are occupied and

b. the portion of the parking spaces that are available.

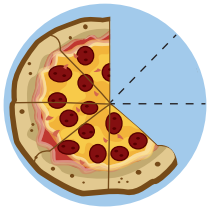


Example 2 Understanding Fractions

Write a fraction indicating

a. the remaining portion of the pizza and

b. the missing portion of the pizza.



Solution

a. The pizza was cut into 8 equal pieces. The 5 pieces remaining represent $\frac{5}{8}$ of the pizza.

b. The missing portion of the pizza represents $\frac{3}{8}$ of the pizza.

Now work margin exercise 2.

Proper Fractions and Improper Fractions

A **proper fraction** is a fraction in which the numerator is less than the denominator. (Proper fractions have values less than 1.)

Examples of proper fractions: $\frac{2}{3}$, $\frac{7}{8}$, and $\frac{32}{60}$

An **improper fraction** is a fraction in which the numerator is greater than or equal to the denominator. (Improper fractions have values greater than or equal to 1.)

Examples of improper fractions: $\frac{15}{8}$, $\frac{14}{14}$, and $\frac{250}{100}$

DEFINITION

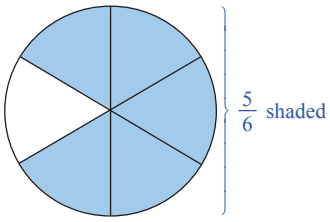
3. Draw a figure to represent the fraction $\frac{3}{8}$.

Example 3 Understanding Proper Fractions

Draw a figure to represent the fraction $\frac{5}{6}$.

Solution

$\frac{5}{6}$ indicates 5 of 6 equal parts. Drawing a figure to represent this fraction, we divide a circle into 6 equal sections and shade 5 of them. (**Note:** Figures other than circles can be used.)

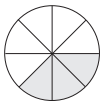


Now work margin exercise 3.

Answers

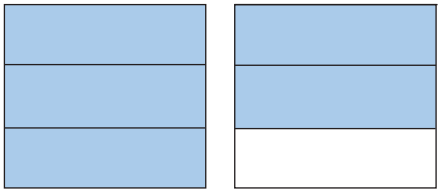
2. a. $\frac{7}{15}$ **b.** $\frac{8}{15}$

3.



Example 4 Understanding Improper Fractions

Write a fraction that indicates the shaded parts of the figure.

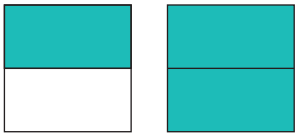


Solution

There are two squares, each separated into 3 equal parts. This means that the denominator is 3. The shading here indicates 5 of these equal parts, which means the numerator is 5. Thus, the shaded part of the figure can be represented by the improper fraction $\frac{5}{3}$.

Now work margin exercise 4.

4. Write a fraction that indicates the shaded parts of the figure.



Whole numbers can be thought of as fractions with denominator 1. Thus, in fraction form

$$0 = \frac{0}{1}, \qquad 1 = \frac{1}{1}, \qquad 2 = \frac{2}{1}, \qquad 3 = \frac{3}{1}, \qquad \text{and so on.}$$

Fraction notation indicates division. For example, $24 \div 8$ can be written in the fraction form $\frac{24}{8}$, which indicates that the numerator is to be divided by the denominator. Thus,

$$\frac{24}{8} = 3, \qquad \frac{45}{5} = 9, \qquad \text{and} \qquad \frac{0}{5} = 0.$$

To discuss fractions in general we need the concept of a variable.

Variable

A **variable** is a symbol (generally a letter of the alphabet) that is used to represent an unknown number.

DEFINITION

Because we know that division by 0 is **undefined**, no denominator can be 0. Thus, in the fraction form $\frac{a}{b}$, we write $b \neq 0$ (read, "b is not equal to 0").

The Number 0 in Fractions

For any nonzero value of b, $\frac{0}{b} = \mathbf{0}$.

For any value of a, $\frac{a}{0}$ is **undefined**.

DEFINITION

Answers

4. $\frac{3}{2}$ (3 of the equal parts)

5. Find the value of each expression.

a. $\frac{0}{45}$

b. $\frac{10}{0}$

Example 5 Evaluating Fractions Involving 0

Find the value of each expression.

a. $\frac{0}{36}$

b. $\frac{0}{124}$

c. $\frac{17}{0}$

d. $\frac{1}{0}$

Solution

a. $\frac{0}{36} = 0$

b. $\frac{0}{124} = 0$

c. $\frac{17}{0}$ is undefined

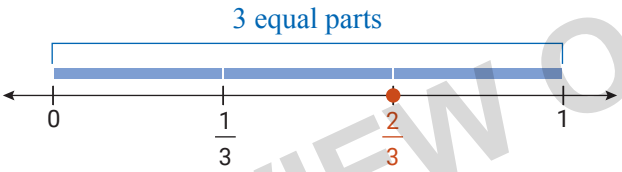
d. $\frac{1}{0}$ is undefined

Now work margin exercise 5.

B Graphing Fractions on a Number Line

We have seen how to "picture" fractions as parts of a whole: part of a whole candy bar, part of a pizza, or part of a shaded region of a whole figure. Another way to visualize a fraction is to mark a corresponding point on a number line. For example, to graph the fraction $\frac{2}{3}$ proceed as follows.

1. Divide the interval (distance) from 0 to 1 into 3 equal parts.
2. Graph (or shade) the second mark to the right of 0.



6. Graph each proper fraction on a number line.

a. $\frac{3}{4}$

b. $\frac{1}{5}$

c. $\frac{1}{2}$

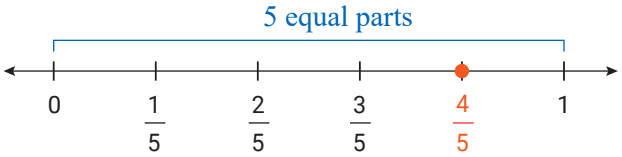
Example 6 Graphing Proper Fractions

Graph each proper fraction on a number line.

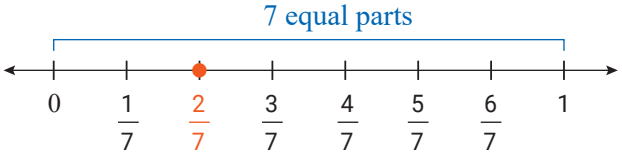
a. $\frac{4}{5}$ **b.** $\frac{2}{7}$ **c.** $\frac{5}{6}$

Solution

a.



b.

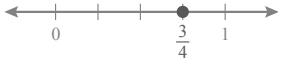


Answers

5. a. 0 **b.** Undefined

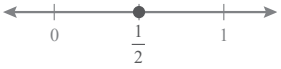
6.

a.

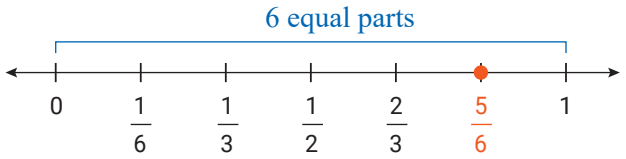


b.

c.



c.



Now work margin exercise 6.

We know that with improper fractions the numerator is equal to or greater than the denominator. This means that the graph of an improper fraction will be at 1 or to the right of 1 on a number line.

Example 7 Graphing Improper Fractions

Graph each of the following improper fractions on a number line.

a. $\frac{7}{5}$ **b.** $\frac{13}{8}$

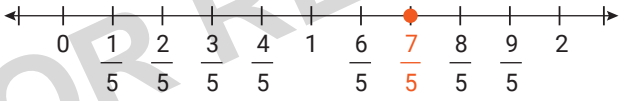
7. Graph each improper fraction on a number line.

a. $\frac{4}{3}$

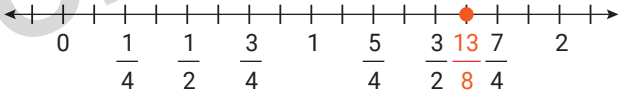
b. $\frac{9}{4}$

Solution

a.



b.



Now work margin exercise 7.

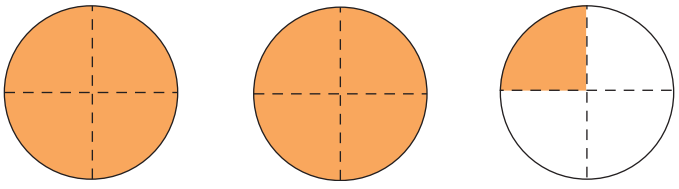
C Introduction to Mixed Numbers

A **mixed number** is the sum of a whole number and a proper fraction. By convention, we usually write the whole number and the fraction side by side without the plus sign. For example, $6+\frac{3}{5}=6\frac{3}{5}$ (read "six and three-fifths").

Typically, people are familiar with mixed numbers and use them frequently. For example, a carpenter might measure a board to be $2\frac{1}{4}$ feet long, or an architect might want to shade $2\frac{1}{4}$ circles in a drawing. A related question would be how many fourths (quarters) of a circle would be shaded? As shown in Figure 3, nine-fourths would be shaded. Thus, the form of an answer might be written as an improper fraction or a mixed number.

$$\overbrace{2\frac{1}{4}}^{\text{mixed number}} = \text{whole number}\ \frac{\text{numerator}}{\text{denominator}}$$

Terminology for mixed numbers

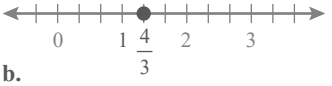


$2\frac{1}{4}=\frac{9}{4}$ shaded

Figure 3

Answers

7. a.



b.

8. Identify each number as a proper fraction, an improper fraction, or a mixed number.

a. $5\frac{2}{3}$

b. $\frac{3}{2}$

c. $\frac{2}{3}$

Example 8 Identifying Types of Fractions and Mixed Numbers

Identify each number as a proper fraction, an improper fraction, or a mixed number.

a. $\frac{8}{7}$ **b.** $1\frac{6}{7}$ **c.** $\frac{7}{8}$

Solution

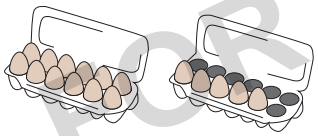
a. improper fraction **b.** mixed number **c.** proper fraction

Now work margin exercise 8.

9. Cassandra has the following eggs in her refrigerator.

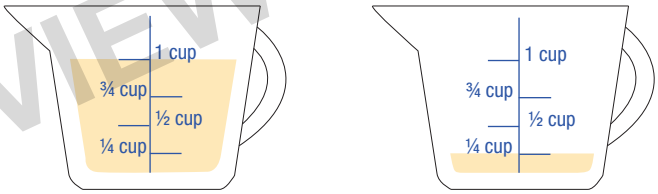
a. Write a mixed number indicating how many cartons of eggs she has (12 eggs equals 1 carton).

b. Write this amount as an improper fraction.



Example 9 Application: Understanding Mixed Numbers

A recipe calls for the amount of oil indicated in the figure.



a. Write a mixed number indicating the amount of oil in the measuring cups.

b. Write this amount as an improper fraction.

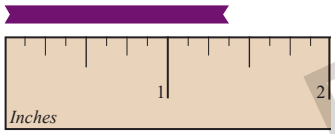
Solution

a. Each cup is marked in fourths and we see that there is a total of $1\frac{1}{4}$ cups.

b. As an improper fraction, $1\frac{1}{4}$ cups $= \frac{5}{4}$ cups.

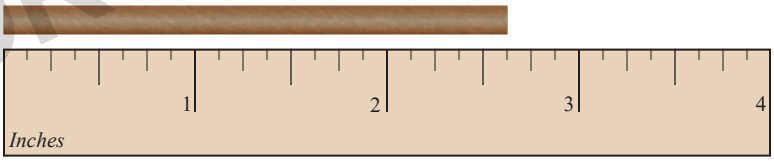
Now work margin exercise 9.

10. A ribbon is cut to the length indicated in the figure. Write the length of the ribbon as a mixed number.



Example 10 Application: Understanding Mixed Numbers

A wooden rod is cut to the length indicated in the figure. Write the length of the rod as a mixed number.



Solution

The ruler is marked in eighths of an inch. The rod measures $2\frac{5}{8}$ in.

Now work margin exercise 10.

Answers

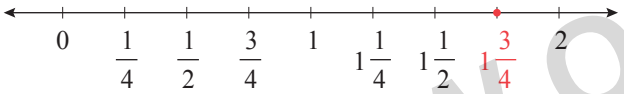
8. a. Mixed number
b. Improper fraction
c. Proper fraction

9. a. $1\frac{5}{12}$ **b.** $\frac{17}{12}$ **10.** $1\frac{3}{8}$ in.

D Graphing Mixed Numbers on a Number Line

We know that mixed numbers are greater than or equal to 1 with a whole number part and a fraction part. To graph a mixed number we can proceed as with fractions by making marks on a number line that correspond to the denominator of the fraction part. For example, to graph the mixed number $1\frac{3}{4}$ proceed as follows.

1. Mark the intervals from 0 to 1 and from 1 to 2 into 4 equal parts.
2. Graph (or shade) the mark at $1\frac{3}{4}$.



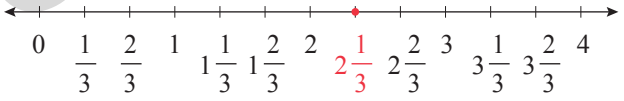
Example 11 Graphing Mixed Numbers

Graph each of the following mixed numbers on a number line.

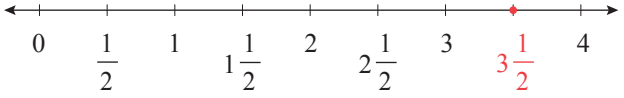
a. $2\frac{1}{3}$ **b.** $3\frac{1}{2}$

Solution

a.



b.



Now work margin exercise 11.

11. Graph each mixed number on a number line.

a. $1\frac{3}{4}$

b. $2\frac{2}{3}$

E Changing Mixed Numbers to Improper Fractions

The following procedure can be used to change mixed numbers to improper fractions.

To Change a Mixed Number to an Improper Fraction

1. Multiply the whole number by the denominator of the proper fraction.
2. Add the numerator of the proper fraction to this product.
3. Write this sum over the denominator of the fraction.

1. $2 \cdot 8 = 16$	2. $16 + 7 = 23$	3.
$2\frac{7}{8} =$	$2\frac{7}{8} =$	$\frac{23}{8}$

PROCEDURE

11.

a.

b.

12. Change $10\frac{4}{9}$ to an improper fraction.

Example 12 Changing Mixed Numbers to Improper Fractions

Change $8\frac{9}{10}$ to an improper fraction.

Solution

Step 1: Multiply the whole number by the denominator: $8 \cdot 10 = 80$

Step 2: Add the numerator: $80 + 9 = 89$

Step 3: Write this sum over the denominator: $8\frac{9}{10} = \frac{89}{10}$

Now work margin exercise 12.

13. Change $3\frac{5}{9}$ to an improper fraction.

Completion Example 13 Mixed Numbers to Improper Fractions

Change $11\frac{2}{3}$ to an improper fraction.

Solution

Step 1: Multiply the whole number by the denominator: $11 \cdot 3 =$ _____

Step 2: Add the numerator: _____ + _____ = _____

Step 3: Write this sum over the denominator: $11\frac{2}{3} = \frac{\_\_\_\_}{\_\_\_\_}$

Now work margin exercise 13.

F Changing Improper Fractions to Mixed Numbers

To reverse the process (that is, to change an improper fraction to a mixed number), we use the fact that a fraction can indicate division.

To Change an Improper Fraction to a Mixed Number

1. Divide the numerator by the denominator. The quotient is the whole number part of the mixed number.
2. Write the remainder over the denominator as the fraction part of the mixed number.

PROCEDURE

Answers

12. $\frac{94}{9}$

13. $\frac{32}{9}$

Completion Example Answer

13. $11 \cdot 3 = 33$; $33 + 2 = 35$; $11\frac{2}{3} = \frac{35}{3}$

Example 14 Changing Improper Fractions to Mixed Numbers

Change $\frac{67}{5}$ to a mixed number.

Solution

Step 1: Divide the numerator by the denominator. The quotient is the whole number part of the mixed number.

$$\begin{array}{r} 13 \\ 5\overline{)67} \\ \underline{5} \\ 17 \\ \underline{15} \\ 2 \end{array}$$

whole number part

remainder

Step 2: Write the remainder over the denominator as the fraction part of the mixed number:

$$\frac{67}{5} = 13 + \frac{2}{5} = 13\frac{2}{5}.$$

Now work margin exercise 14.

14. Change $\frac{92}{11}$ to a mixed number.

Example 15 Changing Improper Fractions to Mixed Numbers

Change $\frac{85}{2}$ to a mixed number.

Solution

Step 1: Divide the numerator by the denominator. The quotient is the whole number part of the mixed number.

$$\begin{array}{r} 42 \\ 2\overline{)85} \\ \underline{8} \\ 05 \\ \underline{4} \\ 1 \end{array}$$

whole number part

remainder

Step 2: Write the remainder over the denominator as the fraction part of the mixed number:

$$\frac{85}{2} = 42 + \frac{1}{2} = 42\frac{1}{2}.$$

Now work margin exercise 15.

15. Change $\frac{77}{66}$ to a mixed number.

CALCULATORS

Using a Calculator to Convert Between Fractions and Mixed Numbers

Many scientific calculators have a fraction button, [a b/c]. To enter a mixed number or fraction in your calculator using this button, press [a b/c] between the whole number and the numerator, and again between the numerator and denominator. (To enter a fraction, simply press [a b/c] between the numerator and denominator.)

If your calculator includes this feature, then you can use it to convert between improper fractions and mixed numbers. Consider the mixed number $2\frac{1}{3}$. To enter this mixed number in your calculator, press the keys

[2] [a b/c] [1] [a b/c] [3].

The calculator will display this as 2_1_3 which means $2\frac{1}{3}$.

Answers

14. $8\frac{4}{11}$ **15.** $1\frac{1}{6}$

To convert this to an improper fraction press [2nd] [a b/c] (or [SHIFT] [a b/c]). This accesses the A b/c ◁▷ d/e feature that converts mixed numbers to fractions and vice versa. The display will read 7_⌋3 which means $\frac{7}{3}$.

Similarly, to convert $\frac{5}{4}$ to a mixed number, press [5] [a b/c] [4] [=]. The display will read 1_1_⌋4 which means $1\frac{1}{4}$.

2.1 Exercises

Concept Check

Fill-in-the-Blank. Complete each sentence using information found in this section.

1. Two fractions that have different denominators but the same value are said to be _____ fractions.
2. If a fraction has a numerator that is equal to or larger than the denominator, it is a/an _____ fraction.
3. A fraction that has a zero in the denominator is considered to be _____.
4. One recommended first step in reducing a fraction is to factor both the numerator and denominator into _____ factors.
5. The sum of a whole number and a proper fraction is called a _____ number.
6. The first step in changing an improper fraction into a mixed number is to divide the _____ by the _____.

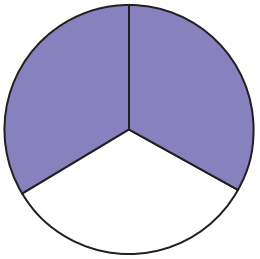
True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

7. Equivalent fractions can be found by multiplying the original numerator and denominator by the same number (or a fraction equivalent to 1).
8. In $\frac{11}{13}$, the denominator is 11.
9. $\frac{0}{6} = 0$
10. $\frac{17}{0}$ is undefined.

Practice

For each figure, write a fraction indicating **a.** the shaded part of the figure and **b.** the unshaded part of the figure. See Examples 1 and 2.

1.



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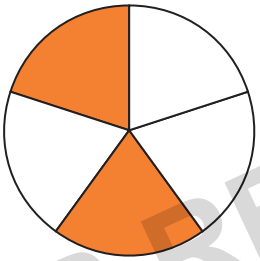

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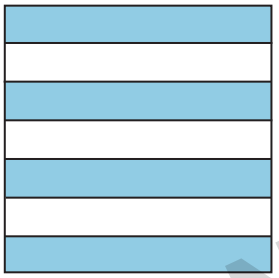

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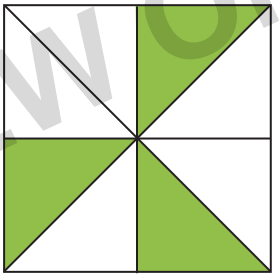
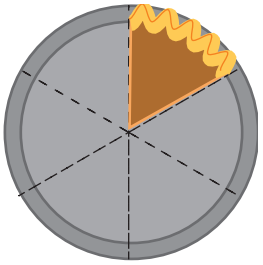


For each figure, write a fraction indicating **a.** the remaining portion of the object and **b.** the missing portion of the object.

5.



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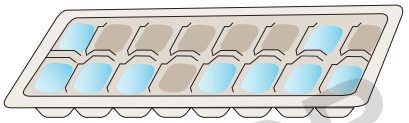
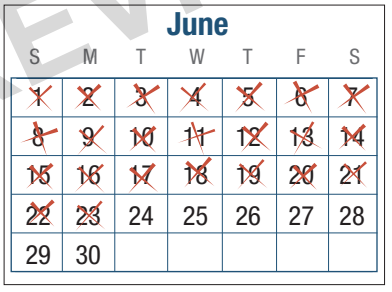


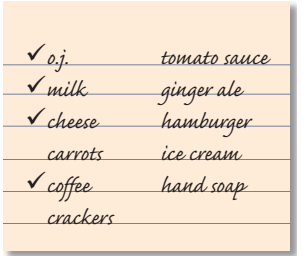
7.

8.

9.



10.

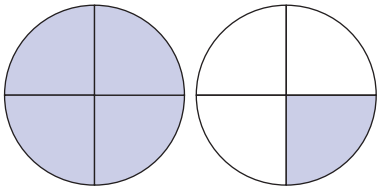


Draw a figure to represent each fraction. See Example 3.

11. $\frac{1}{3}$ **12.** $\frac{1}{2}$ **13.** $\frac{4}{5}$ **14.** $\frac{3}{4}$

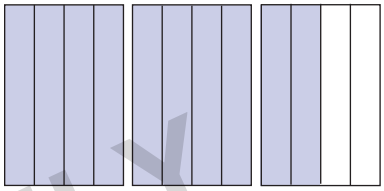
Write a fraction that indicates the shaded parts of each figure. See Example 4.

15.

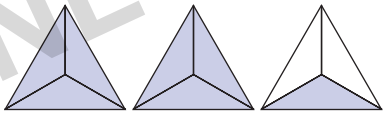


16.

17.



18.



Find the value of each expression. See Example 5.

19. $\frac{0}{6}$ **20.** $\frac{0}{35}$ **21.** $\frac{15}{0}$ **22.** $\frac{2}{0}$

Graph each fraction on a number line. See Examples 6 and 7.

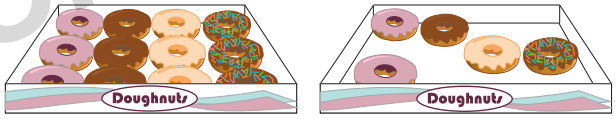
23. $\frac{3}{5}$ **24.** $\frac{3}{8}$ **25.** $\frac{6}{5}$ **26.** $\frac{8}{3}$

Identify each number as a proper fraction, an improper fraction, or a mixed number. See Example 8.

27. $1\frac{1}{2}$ **28.** $\frac{5}{3}$ **29.** $\frac{7}{8}$ **30.** $7\frac{5}{12}$

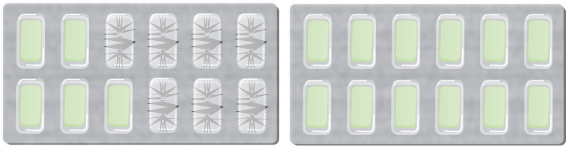
Write each amount described as **a.** a mixed number and **b.** an improper fraction. See Example 9.

31. Isabella brought 2 boxes of donuts to a meeting. The figure shows the remaining amount of donuts.

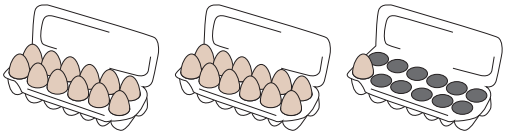


32. A recipe calls for the amount of tomato juice indicated in the figure.

33. Shane has two blister packs of gum. The figure shows the remaining amount of gum.

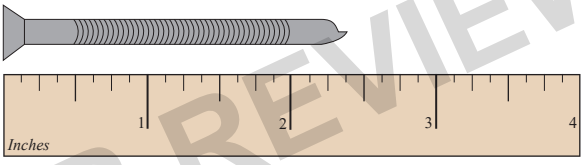


34. Cassandra has the following eggs in her refrigerator.



Write a mixed number to describe the length indicated in each figure. See Example 10.

35.



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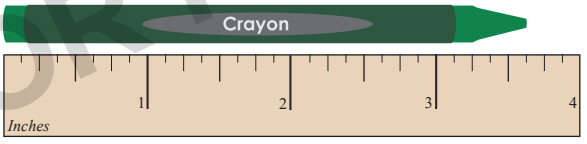

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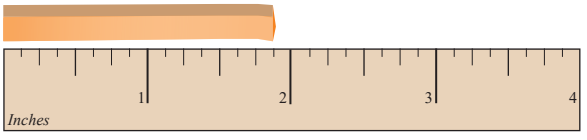

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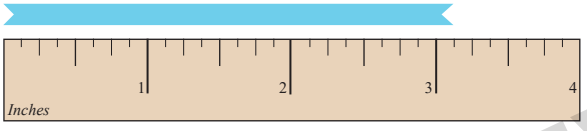


Graph each mixed number on a number line. See Example 11.

39. $1\frac{1}{3}$ **40.** $3\frac{1}{4}$ **41.** $2\frac{2}{5}$ **42.** $1\frac{3}{8}$

Change each mixed number to an improper fraction. See Examples 12 and 13.

43. $1\frac{3}{5}$ **47.** $6\frac{8}{10}$ **51.** $4\frac{6}{7}$ **55.** $10\frac{8}{12}$

44. $1\frac{2}{3}$ **48.** $9\frac{4}{10}$ **52.** $7\frac{1}{7}$ **56.** $3\frac{1}{50}$

45. $2\frac{1}{4}$ **49.** $4\frac{5}{8}$ **53.** $1\frac{2}{15}$ **57.** $7\frac{1}{100}$

46. $4\frac{3}{4}$ **50.** $6\frac{4}{8}$ **54.** $1\frac{3}{17}$ **58.** $6\frac{19}{100}$

Change each improper fraction to a mixed number. See Examples 14 and 15.

59. $\frac{4}{3}$

60. $\frac{11}{8}$

61. $\frac{13}{2}$

62. $\frac{19}{4}$

63. $\frac{5}{2}$

64. $\frac{17}{8}$

65. $\frac{27}{10}$

66. $\frac{33}{10}$

67. $\frac{37}{8}$

68. $\frac{29}{6}$

69. $\frac{36}{12}$

70. $\frac{48}{16}$

71. $\frac{35}{20}$

72. $\frac{14}{5}$

73. $\frac{185}{100}$

74. $\frac{329}{100}$

Applications

Solve.

75. ***Eating Out:*** If you had $20 and you spent $9 for a hamburger, fries, and a soft drink, what fraction of your money did you spend? What fraction would you still have?

76. ***Grades:*** In a class of 35 students, 6 students received As on a mathematics exam. What fraction of students received an A? What fraction of students did not receive an A?

77. ***Customer Support:*** A software company receives 45 technical support calls in one hour. Twenty-three of the calls are related to a customer forgetting their password. What fraction of the calls was related to a customer forgetting their password?

78. ***Nutrition:*** A certain brand of plain bagels has 146 calories per bagel. 115 calories come from the carbohydrates in the bagel. What fraction of the calories is from carbohydrates?

79. ***Time:*** What fraction of a minute does 43 seconds represent? (**Hint:** There are 60 seconds in a minute.)

80. ***Distance:*** There are 5280 feet in a mile. What fraction of a mile does 923 feet represent?

81. ***Computers:*** A computer stores data on a hard drive in the form of bits, bytes, and sectors.

a. Each byte is made up of eight bits. What fraction of a byte is a bit?

b. A sector on a hard drive is traditionally 512 bytes. A byte is what fraction of a sector?

c. If a computer stores 159 bytes of data, what fraction of a sector does that amount of data take up?

82. ***Travel:*** The gas tank of a car holds 14 gallons of gas. What fraction of the tank does 9 gallons of gas take up?

83. ***Shipping:*** A small box will hold 12 books. Kathleen has 35 books to pack into small boxes.

a. Write an improper fraction to describe the number of boxes that will be filled by Kathleen's books.

b. Change the improper fraction from part a. to a mixed number to describe the number of boxes that will be filled by Kathleen's books.

84. ***Beverages:*** A cup holds 8 ounces of liquid. You have 29 ounces of juice to pour into cups.

a. Write an improper fraction to describe the number of cups that will be filled with the juice.

b. Change the improper fraction from part a. to a mixed number to describe the number of cups that will be filled with the juice.

Writing & Thinking

85. In your own words, list the parts of a fraction and briefly describe the purpose of each part.

86. Give an example of a situation where you might use fractions and/or mixed numbers outside of class.

87. Show and explain, using diagrams and words, why $2\frac{3}{5} = \frac{13}{5}$.

88. Explain how to change an improper fraction into a mixed number.

Objectives

A. Multiply fractions.

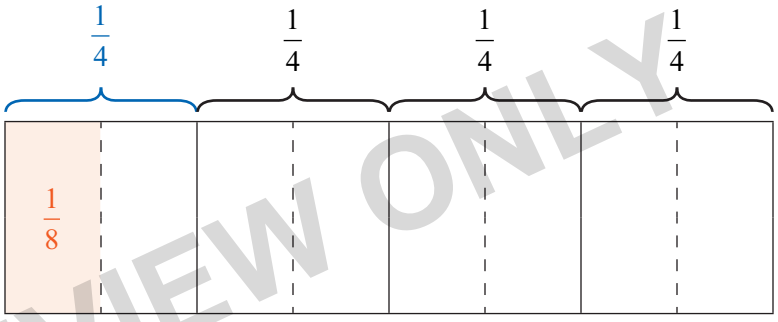
B. Reduce fractions to lowest terms.

C. Multiply and reduce fractions to lowest terms.

2.2 Multiplication with Fractions

A Multiplication with Fractions

Finding the product of two fractions can be thought of as finding one fractional part of another fraction. For example, when we multiply $\frac{1}{2}$ by $\frac{1}{4}$ we are finding $\frac{1}{2}$ **of** $\frac{1}{4}$.



$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$

Figure 1

Now we state the rule for multiplying fractions. Remember that any whole number can be written in fraction form with a denominator of 1 and no denominator can be 0.

To Multiply Fractions

1. Multiply the numerators.
2. Multiply the denominators.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \quad (b, d \neq 0)$$

Example: $\frac{1}{2} \cdot \frac{3}{7} = \frac{1 \cdot 3}{2 \cdot 7} = \frac{3}{14}$

PROCEDURE

1. Find $\frac{3}{4}$ of $\frac{5}{8}$.

Example 1 Multiplying Fractions

Find $\frac{2}{5}$ of $\frac{7}{3}$.

Solution

Remember, to find a fraction **of** a number means to multiply the number by the fraction.

$$\frac{2}{5} \cdot \frac{7}{3} = \frac{2 \cdot 7}{5 \cdot 3} = \frac{14}{15}$$

Now work margin exercise 1.

Answers

1. $\frac{15}{32}$

Example 2 Multiplying Fractions

Multiply.

a. $\frac{6}{7}\cdot\frac{8}{5}$ **b.** $\frac{4}{13}\cdot 3$ **c.** $\frac{9}{8}\cdot 0$ **d.** $\frac{4}{3}\cdot\frac{5}{3}\cdot\frac{2}{7}$

Solution

a. $\frac{6}{7}\cdot\frac{8}{5}=\frac{6\cdot 8}{7\cdot 5}=\frac{48}{35}$ or $1\frac{13}{35}$

b. $\frac{4}{13}\cdot 3=\frac{4}{13}\cdot\frac{3}{1}=\frac{4\cdot 3}{13\cdot 1}=\frac{12}{13}$

c. $\frac{9}{8}\cdot 0=\frac{9}{8}\cdot\frac{0}{1}=\frac{9\cdot 0}{8\cdot 1}=\frac{0}{8}=0$

d. $\frac{4}{3}\cdot\frac{5}{3}\cdot\frac{2}{7}=\frac{4\cdot 5\cdot 2}{3\cdot 3\cdot 7}=\frac{40}{63}$

Now work margin exercise 2.

2. Multiply.

a. $\frac{2}{3}\cdot\frac{11}{9}$

b. $\frac{4}{5}\cdot 7$

c. $\frac{0}{4}\cdot\frac{11}{16}$

d. $\frac{1}{7}\cdot\frac{3}{5}\cdot\frac{3}{2}$

Example 3 Application: Multiplying Fractions

In a certain voting district, $\frac{3}{5}$ of the eligible voters are actually registered to vote. Of those registered voters, $\frac{2}{7}$ are independents (have no party affiliation). What fraction of the eligible voters are registered independents?

Solution

Step 1: READ: Read the problem carefully. There are two types of voters to consider: registered voters and independent voters.

Step 2: SET UP: In looking for the fraction of registered independent voters, we know that $\frac{2}{7}$ of $\frac{3}{5}$ of eligible voters are registered independents, so will need to multiply $\frac{2}{7}\cdot\frac{3}{5}$.

Step 3: SOLVE: $\frac{2}{7}\cdot\frac{3}{5}=\frac{2\cdot 3}{7\cdot 5}=\frac{6}{35}$

Thus, $\frac{6}{35}$ of the eligible voters are registered as independents.

Step 4: CHECK: The fraction $\frac{3}{5}$ is a little more than $\frac{1}{2}$, which means that approximately $\frac{1}{7}$ of eligible voters are registered as independents. Since $\frac{1}{7}=\frac{5}{35}$, the fraction $\frac{6}{35}$ seems like a reasonable answer as it is a little more than $\frac{5}{35}$.

Now work margin exercise 3.

3. For a certain piece of furniture, $\frac{1}{8}$ of the included hardware is wing nuts. Of these wing nuts, $\frac{1}{20}$ are defective. What fraction of the included wingnuts are defective?

Both the commutative property and the associative property of multiplication apply to fractions.

Answers

2. a. $\frac{22}{27}$ **b.** $\frac{28}{5}$ or $5\frac{3}{5}$ **c.** 0 **d.** $\frac{9}{70}$

3. $\frac{1}{160}$

Commutative Property of Multiplication

The **order** of the fractions being multiplied can be reversed without changing the product. Symbolically, if $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{c}{d}\cdot\frac{a}{b}. \quad (b,d\neq 0)$$

For example, $\frac{2}{5}\cdot\frac{1}{3}=\frac{1}{3}\cdot\frac{2}{5}$.

PROPERTIES

Associative Property of Multiplication

The **grouping** of the fractions being multiplied can be changed without changing the product. Symbolically, if $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are fractions, then

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f}=\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right). \quad (b,d,f\neq 0)$$

For example, $\left(\frac{5}{6}\cdot\frac{2}{3}\right)\cdot\frac{1}{4}=\frac{5}{6}\cdot\left(\frac{2}{3}\cdot\frac{1}{4}\right)$.

PROPERTIES

4. For each statement, state the property of multiplication illustrated.

a. $\frac{1}{3}\cdot\left(\frac{1}{2}\cdot\frac{1}{4}\right)=\left(\frac{1}{3}\cdot\frac{1}{2}\right)\cdot\frac{1}{4}$

b. $\frac{1}{2}\cdot\frac{5}{8}=\frac{5}{8}\cdot\frac{1}{2}$

Example 4 Recognizing the Properties of Multiplication

Each of the properties of multiplication is illustrated.

a. $\frac{3}{4}\cdot\frac{1}{7}=\frac{1}{7}\cdot\frac{3}{4}$ Commutative property of multiplication

As a check, we see that $\frac{3}{4}\cdot\frac{1}{7}=\frac{3}{28}$ and $\frac{1}{7}\cdot\frac{3}{4}=\frac{3}{28}$.

b. $\left(\frac{5}{8}\cdot\frac{3}{2}\right)\cdot\frac{3}{4}=\frac{5}{8}\cdot\left(\frac{3}{2}\cdot\frac{3}{4}\right)$ Associative property of multiplication

As a check, we see that

$$\left(\frac{5}{8}\cdot\frac{3}{2}\right)\cdot\frac{3}{4}=\frac{15}{16}\cdot\frac{3}{4}=\frac{45}{64} \quad \text{and} \quad \frac{5}{8}\cdot\left(\frac{3}{2}\cdot\frac{3}{4}\right)=\frac{5}{8}\cdot\frac{9}{8}=\frac{45}{64}.$$

Now work margin exercise 4.

Answers

4. a. Associative property of multiplication **b.** Commutative property of multiplication

B Reducing Fractions to Lowest Terms

Figure 2 shows how a whole can be separated into equal parts in several ways. The shaded parts represent the same "fraction" of the whole.

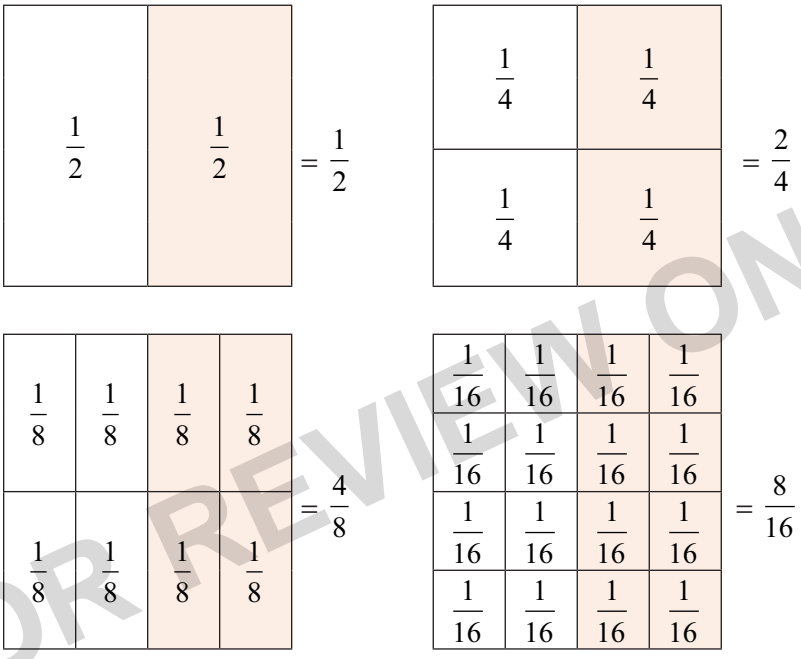


Figure 2

From the shading in Figure 2, we see that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$. These fractions are **equal fractions**. Equal fractions are said to be **equivalent**.

Finding an equivalent fraction in lower terms (with a smaller denominator) is called **reducing** the fraction. **A fraction is reduced to lowest terms if the numerator and denominator have no common factors other than 1.**

The use of prime factors is very helpful in reducing to lowest terms.

To Reduce a Fraction to Lowest Terms

1. Factor the numerator and denominator into prime factors.
2. Use the fact that $\frac{k}{k} = 1$ and "divide out" all common factors.

Note: Reduced fractions may be improper fractions.

PROCEDURE

5. Reduce each fraction to lowest terms.

a. $\frac{8}{36}$

b. $\frac{49}{28}$

Example 5 Reducing Fractions to Lowest Terms

Reduce each fraction to lowest terms.

a. $\frac{15}{20}$ **b.** $\frac{35}{21}$

Solution

a. $\frac{15}{20} = \frac{3 \cdot 5}{2 \cdot 2 \cdot 5} = \frac{3}{2 \cdot 2} \cdot \frac{5}{5} = \frac{3}{4} \cdot 1 = \frac{3}{4}$

b. $\frac{35}{21} = \frac{5 \cdot 7}{3 \cdot 7} = \frac{5}{3} \cdot \frac{7}{7} = \frac{5}{3} \cdot 1 = \frac{5}{3}$

Note that the rules for divisibility quickly indicate that 5 is a factor of 15, 20, and 35, and 3 is a factor of 21.

Now work margin exercise 5.

As a shortcut, we do not generally write the number 1 when reducing the form $\frac{k}{k}$, as we did in Example 5. Instead, we use cancel marks to indicate dividing out common factors in numerators and denominators. **But remember that these numbers do not simply disappear. Their quotient is understood to be 1 even if the 1 is not written.**

6. Reduce $\frac{16}{64}$ to lowest terms.

Example 6 Reducing Fractions to Lowest Terms

Reduce $\frac{8}{72}$ to lowest terms.

Solution

Remember that 1 is a factor of any whole number. So, if all of the factors in the numerator or denominator are divided out, 1 must be used as a factor.

Using prime factors, we get the following.

$$\frac{8}{72} = \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 1}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 3 \cdot 3} = \frac{1}{9}$$

Or, if you see that 8 is a common factor, divide it out. But remember that 1 is a factor.

$$\frac{8}{72} = \frac{\cancel{8} \cdot 1}{\cancel{8} \cdot 9} = \frac{1}{9}$$

Now work margin exercise 6.

Example 6 illustrated the fact that when reducing, common factors in the numerator and denominator need not be prime. However, the advantage of using prime factors is that you can be certain that the fraction is reduced to lowest terms.

Answers

5. a. $\frac{2}{9}$ **b.** $\frac{7}{4}$

6. $\frac{1}{4}$

Completion Example 7 Reducing Fractions to Lowest Terms

Reduce $\frac{52}{65}$ to lowest terms.

Solution

Finding a common factor could be difficult here. Prime factoring helps.

$$\frac{52}{65} = \frac{2 \cdot 2 \cdot \square}{5 \cdot \square} = \underline{\qquad\qquad}$$

Now work margin exercise 7.

7. Reduce $\frac{48}{66}$ to lowest terms.

Example 8 Application: Reducing Fractions to Lowest Terms

Suppose you had \$25 and you spent \$15 to buy music.

a. What fraction of your money did you spend on music?

b. What fraction do you still have?

Solution

a. The fraction you spent is $\frac{15}{25} = \frac{3 \cdot \cancel{5}}{5 \cdot \cancel{5}} = \frac{3}{5}$.

b. Since you still have \$25 – \$15 = \$10, the fraction you still have is

$$\frac{10}{25} = \frac{2 \cdot \cancel{5}}{5 \cdot \cancel{5}} = \frac{2}{5}.$$

Now work margin exercise 8.

8. Suppose you had \$42 and spent \$30 to buy books.

a. What fraction of your money did you spend on books?

b. What fraction do you still have?

C Multiplying and Reducing Fractions

Now we can multiply fractions and reduce all in one step by using prime factors (or other common factors). If you have any difficulty understanding how to multiply and reduce, use prime factors. By using prime factors, you can be sure that you have not missed a common factor and that your answer is reduced to lowest terms.

Examples 9 through 12 illustrate how to multiply and reduce at the same time by factoring the numerators and the denominators. Note that **if all the factors in the numerator or denominator divide out, then 1 must be used as a factor**. (See Examples 11 and 12.)

CALCULATORS

Using a Calculator to Reduce Fractions

To reduce fractions on a calculator, you will need the fraction button, [a b/c]. Reduce a fraction such as $\frac{12}{15}$ by pressing the keys [1] [2] [a b/c] [1] [5]. Then press [=].

The calculator will display 4_|5 or $\frac{4}{5}$, the simplified form of the fraction.

Completion Example Answers

7. $\frac{2 \cdot 2 \cdot \cancel{13}}{5 \cdot \cancel{13}} = \frac{4}{5}$

Answers

7. $\frac{8}{11}$

8. a. $\frac{5}{7}$ **b.** $\frac{2}{7}$

9. Multiply and reduce to lowest terms:
$\frac{4}{15}\cdot\frac{35}{48}$

Example 9 Multiplying and Reducing Using Prime Factors

Multiply and reduce to lowest terms: $\frac{15}{28}\cdot\frac{7}{9}$

Solution

Do not start by multiplying the numerators and denominators. The results would simply be large numbers that would then need to be factored. Using prime factors, we have

$$\frac{15}{28}\cdot\frac{7}{9}=\frac{15\cdot 7}{28\cdot 9}=\frac{\cancel{3}\cdot 5\cdot\cancel{7}}{2\cdot 2\cdot\cancel{7}\cdot\cancel{3}\cdot 3}=\frac{5}{2\cdot 2\cdot 3}=\frac{5}{12}.$$

Note that we did not find the product in the numerator or denominator before factoring.

Now work margin exercise 9.

10. Multiply and reduce to lowest terms:
$\frac{5}{6}\cdot\frac{8}{7}\cdot\frac{14}{10}$

Example 10 Multiplying and Reducing Using Prime Factors

Multiply and reduce to lowest terms: $\frac{9}{10}\cdot\frac{25}{32}\cdot\frac{44}{33}$

Solution

Using prime factors, we have

$$\frac{9}{10}\cdot\frac{25}{32}\cdot\frac{44}{33}=\frac{9\cdot 25\cdot 44}{10\cdot 32\cdot 33}=\frac{\cancel{3}\cdot 3\cdot\cancel{5}\cdot 5\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{11}}{2\cdot\cancel{5}\cdot 2\cdot 2\cdot 2\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{11}}=\frac{3\cdot 5}{2\cdot 2\cdot 2\cdot 2}=\frac{15}{16}.$$

Now work margin exercise 10.

11. Multiply and reduce to lowest terms:
$16\cdot\frac{12}{100}\cdot\frac{5}{36}$

Example 11 Multiplying and Reducing Using Prime Factors

Multiply and reduce to lowest terms: $\frac{17}{50}\cdot\frac{25}{34}\cdot 8$

Solution

$$\frac{17}{50}\cdot\frac{25}{34}\cdot 8=\frac{17\cdot 25\cdot 8}{50\cdot 34\cdot 1}$$

Note that $8=\frac{8}{1}$.

$$=\frac{\cancel{17}\cdot\cancel{25}\cdot\cancel{2}\cdot\cancel{2}\cdot 2}{\cancel{2}\cdot\cancel{25}\cdot\cancel{2}\cdot\cancel{17}\cdot 1}$$

In this example, 25 is a common factor that is not a prime number.

$$=\frac{2}{1}=2$$

Now work margin exercise 11.

Answers

9. $\frac{7}{36}$

10. $\frac{4}{3}$ or $1\frac{1}{3}$

11. $\frac{4}{15}$

Completion Example 12 Multiplying and Reducing Using Prime Factors

Multiply and reduce to lowest terms: $\frac{55}{26}\cdot\frac{8}{44}\cdot\frac{91}{35}$

Solution

$$\frac{55}{26}\cdot\frac{8}{44}\cdot\frac{91}{35}=\frac{55\cdot 8\cdot 91}{26\cdot 44\cdot 35}$$

$$=\frac{\_\_\_\_\_}{\_\_\_\_\_}$$

$$=\frac{\_\_\_\_}{\_\_\_\_}$$

$$=\frac{\_\_\_\_}{\_\_\_\_}$$

Now work margin exercise 12.

12. Multiply and reduce to lowest terms:

$\frac{15}{6}\cdot\frac{4}{19}\cdot\frac{38}{10}$

Example 13 Application: Multiplying and Reducing Fractions

A study showed that $\frac{5}{8}$ of the members of a public service organization were in favor of a new set of bylaws. If the organization had a membership of 200 people, how many were in favor of the changes in the bylaws?

Solution

Step 1: READ: Read the problem carefully. Note we need to find a fraction of the membership.

Step 2: SET UP: To find $\frac{5}{8}$ of 200, multiply.

Step 3: SOLVE: $\frac{5}{8}\cdot 200=\frac{5}{8}\cdot\frac{200}{1}=\frac{5\cdot\cancel{8}\cdot 25}{\cancel{8}\cdot 1}=\frac{125}{1}=125$

Thus, there are 125 members in favor of the bylaw changes.

Step 4: CHECK: The fraction $\frac{5}{8}$ is a little more than $\frac{1}{2}$, which gives an approximation of 100 members in favor of the law. Therefore, the answer of 125 members is reasonable as it is a little more than 100.

Now work margin exercise 13.

13. A delivery service found that $\frac{9}{10}$ of its drivers wore their seat belts. If this company employed 500 drivers, how many drivers wore their seat belts?

Another method frequently used to multiply and reduce at the same time is to divide numerators and denominators by common factors whether they are prime or not. If these factors are easily determined, then this method is probably faster. But common factors are sometimes missed with this method whereas they are not missed with the prime factorization method.

Examples 9 through 12 are shown again as Examples 14 through 17, this time using the division method.

Completion Example Answers

12. $\frac{55}{26}\cdot\frac{8}{44}\cdot\frac{91}{35}=\frac{55\cdot 8\cdot 91}{26\cdot 44\cdot 35}=\frac{\cancel{5}\cdot\cancel{11}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{7}\cdot\cancel{13}}{\cancel{2}\cdot\cancel{13}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{11}\cdot\cancel{7}\cdot\cancel{5}}=\frac{1}{1}=1$

Answers

12. 2

13. 450 drivers wore their seat belts

14. Multiply and reduce to lowest terms:

$\frac{4}{15} \cdot \frac{35}{48}$

Example 14 Multiplying and Reducing Using the Division Method

Multiply and reduce to lowest terms: $\frac{15}{28} \cdot \frac{7}{9}$

Solution

$$\frac{\overset{5}{\cancel{15}}}{\underset{4}{\cancel{28}}} \cdot \frac{\overset{1}{\cancel{7}}}{\underset{3}{\cancel{9}}} = \frac{5}{12}$$

3 divides both 15 and 9.
7 divides both 7 and 28.

Now work margin exercise 14.

15. Multiply and reduce to lowest terms:

$\frac{14}{15} \cdot \frac{25}{21} \cdot \frac{3}{10}$

Example 15 Multiplying and Reducing Using the Division Method

Multiply and reduce to lowest terms: $\frac{9}{10} \cdot \frac{25}{32} \cdot \frac{44}{33}$

Solution

$$\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}} \cdot \frac{\overset{5}{\cancel{25}}}{\underset{8}{\cancel{32}}} \cdot \frac{\overset{1}{\cancel{44}}}{\underset{1}{\cancel{33}}} = \frac{15}{16}$$

5 divides both 25 and 10.
11 divides both 44 and 33.
4 divides both 4 and 32.
3 divides both 9 and 3.

Now work margin exercise 15.

16. Multiply and reduce to lowest terms:

$\frac{8}{9} \cdot 15 \cdot \frac{4}{30}$

Example 16 Multiplying and Reducing Using the Division Method

Multiply and reduce to lowest terms: $\frac{17}{50} \cdot \frac{25}{34} \cdot 8$

Solution

$$\frac{\overset{1}{\cancel{17}}}{\underset{1}{\cancel{50}}} \cdot \frac{\overset{1}{\cancel{25}}}{\underset{1}{\cancel{34}}} \cdot \frac{\overset{2}{\cancel{8}}}{1} = \frac{2}{1} = 2$$

17 divides both 17 and 34.
25 divides both 25 and 50.
2 divides both 8 and 2.
2 divides both 4 and 2.

Now work margin exercise 16.

17. Multiply and reduce to lowest terms:

$\frac{17}{100} \cdot \frac{27}{34} \cdot \frac{25}{9}$

Example 17 Multiplying and Reducing Using the Division Method

Multiply and reduce to lowest terms: $\frac{55}{26} \cdot \frac{8}{44} \cdot \frac{91}{35}$

Solution

$$\frac{\overset{1}{\cancel{55}}}{\underset{1}{\cancel{26}}} \cdot \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{44}}} \cdot \frac{\overset{1}{\cancel{91}}}{\underset{1}{\cancel{35}}} = \frac{1}{1} = 1$$

11 divides both 55 and 44. 13 divides both 91 and 26.
5 divides both 5 and 35. 7 divides both 7 and 7.
2 divides both 8 and 2. 4 divides both 4 and 4.

Now work margin exercise 17.

Answer

14. $\frac{7}{36}$

15. $\frac{1}{3}$

16. $\frac{16}{9}$ or $1\frac{7}{9}$

17. $\frac{3}{8}$

2.2 Exercises

Concept Check

Fill-in-the-Blank. Complete each sentence using information found in this section.

1. Any whole number can be written in fraction form with denominator _____.
2. Finding a fraction "of" a number requires _____.
3. To reduce a fraction to lowest terms, divide out any common _____.
4. If all the factors in the numerator or denominator are divided out, then _____ must be used as a factor.
5. To multiply and reduce at the same time, divide numerators and denominators by _____ factors.
6. Finding _____ factorizations may help in multiplying and reducing at the same time.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

7. To find $\frac{1}{2}$ of $\frac{2}{9}$ requires multiplication.
8. $\frac{3}{4} \cdot \frac{9}{10} = \frac{27}{40}$
9. The statement $\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{5} \cdot \frac{1}{3}$ is an example of the associative property of multiplication.
10. The number 1 is always a factor of the numerator and the denominator.

Practice

Multiply. See Examples 1 and 2.

1. $\frac{1}{4} \cdot \frac{1}{4}$
2. $\frac{1}{7} \cdot \frac{1}{7}$
3. $\frac{2}{5} \cdot \frac{2}{5}$
4. $\frac{3}{7} \cdot \frac{3}{7}$
5. $\frac{0}{3} \cdot \frac{5}{7}$
6. $\frac{0}{4} \cdot \frac{7}{6}$
7. $\frac{4}{1} \cdot \frac{3}{1}$
8. $\frac{2}{1} \cdot \frac{5}{1}$
9. $\frac{3}{5} \cdot \frac{4}{7}$
10. $\frac{2}{3} \cdot \frac{5}{11}$
11. $\frac{5}{8} \cdot \frac{3}{4}$
12. $\frac{7}{6} \cdot \frac{5}{2}$
13. $\frac{15}{1} \cdot \frac{3}{2}$
14. $\frac{6}{5} \cdot \frac{7}{1}$
15. $\frac{1}{2} \cdot \frac{3}{7} \cdot \frac{9}{2}$

16. $\frac{7}{3} \cdot \frac{2}{5} \cdot \frac{1}{9}$

17. $\frac{7}{8} \cdot \frac{7}{9} \cdot \frac{7}{3}$

18. $\frac{8}{5} \cdot \frac{8}{5} \cdot \frac{7}{1}$

19. Find $\frac{1}{3}$ of $\frac{1}{2}$.

20. Find $\frac{1}{2}$ of $\frac{1}{3}$.

21. Find $\frac{2}{3}$ of $\frac{2}{15}$.

22. Find $\frac{4}{7}$ of $\frac{3}{5}$.

23. Find $\frac{1}{3}$ of $\frac{2}{3}$.

24. Find $\frac{1}{4}$ of $\frac{3}{4}$.

Reduce each fraction to lowest terms. If it is already in lowest terms, simply rewrite the fraction. See Examples 5 and 6.

25. $\frac{3}{9}$

26. $\frac{2}{8}$

27. $\frac{9}{12}$

28. $\frac{6}{20}$

29. $\frac{5}{11}$

30. $\frac{7}{13}$

31. $\frac{0}{25}$

32. $\frac{0}{16}$

33. $\frac{16}{40}$

34. $\frac{14}{36}$

35. $\frac{12}{35}$

36. $\frac{27}{56}$

37. $\frac{16}{32}$

38. $\frac{25}{50}$

39. $\frac{26}{39}$

40. $\frac{36}{48}$

41. $\frac{42}{63}$

42. $\frac{64}{96}$

43. $\frac{25}{76}$

44. $\frac{12}{35}$

45. $\frac{48}{12}$

46. $\frac{72}{36}$

47. $\frac{24}{100}$

48. $\frac{72}{100}$

49. $\frac{30}{70}$

50. $\frac{70}{100}$

51. $\frac{144}{156}$

52. $\frac{121}{165}$

53. $\frac{150}{135}$

54. $\frac{140}{112}$

55. $\frac{390}{260}$

56. $\frac{560}{420}$

Multiply and reduce to lowest terms. See Examples 9 through 17. (Hint: Factor before multiplying.)

57. $\frac{1}{3} \cdot \frac{3}{4}$

58. $\frac{3}{7} \cdot \frac{5}{3}$

59. $\frac{2}{3} \cdot \frac{4}{3}$

60. $\frac{1}{5} \cdot \frac{3}{5}$

61. $\frac{1}{5} \cdot \frac{4}{7}$

62. $\frac{3}{5} \cdot \frac{2}{7}$

63. $\frac{5}{16} \cdot \frac{16}{15}$

64. $\frac{14}{9} \cdot \frac{3}{14}$

65. $\frac{7}{8} \cdot \frac{9}{14}$

66. $\frac{6}{7} \cdot \frac{5}{12}$

67. $\frac{10}{18} \cdot \frac{9}{5}$

68. $\frac{8}{10} \cdot \frac{5}{4}$

69. $\frac{2}{21} \cdot \frac{15}{22}$

70. $\frac{3}{16} \cdot \frac{20}{21}$

71. $\frac{15}{27} \cdot \frac{9}{30}$

72. $\frac{25}{9} \cdot \frac{3}{100}$

73. $8 \cdot \frac{5}{12}$

74. $9 \cdot \frac{7}{24}$

75. $\frac{32}{20} \cdot \frac{13}{9} \cdot \frac{7}{26}$

76. $\frac{20}{32} \cdot \frac{9}{13} \cdot \frac{26}{7}$

77. $\frac{9}{10} \cdot \frac{35}{40} \cdot \frac{25}{15}$

78. $\frac{5}{12} \cdot \frac{56}{42} \cdot \frac{90}{54}$

79. $\frac{17}{100} \cdot \frac{27}{34} \cdot \frac{25}{9} \cdot 6$

80. $\frac{13}{28} \cdot \frac{7}{9} \cdot \frac{45}{39} \cdot 4$

Applications

Solve.

81. ***Food Prep:*** A pizza is to be cut into fourths. Each of these fourths is to be cut into thirds. What fraction of the pizza is each of the final pieces?

82. ***Food:*** One of Maria's birthday presents was a box of candy. Half of the candy was chocolate covered and one-fourth of the chocolate covered candy had cherries inside. What fraction of the candy were chocolate covered cherries?

83. ***Recipes:*** A recipe calls for $\frac{3}{4}$ cup of flour. How much flour should be used if only half of the recipe is to be made?

84. ***Inventory:*** In a box of mixed colored ink cartridges for a desk printer, $\frac{5}{6}$ of the cartridges are not black ink. Of these nonblack ink cartridges, $\frac{1}{2}$ are magenta. What fraction of the cartridges is magenta?

85. ***Education:*** Of the books in a personal library, $\frac{3}{4}$ are fiction. Of these books, $\frac{3}{5}$ are paperback. What fraction of the books in the library are fiction and paperbacks?

86. ***Technology:*** While shopping for a new smartphone, Jasmine found that $\frac{2}{3}$ of the smartphones on the market have the ability to record high-definition videos. Of these smartphones, $\frac{2}{5}$ have 64 GB of memory. What fraction of the smartphones on the market can take high-definition video and have 64 GB of memory?

87. ***Beverages:*** A glass is 8 inches tall. If the glass is $\frac{3}{4}$ full of water, what is the height of the water in the glass?

88. ***Demographics:*** A study showed that $\frac{3}{5}$ of the students in an elementary school were left-handed. If the school had an enrollment of 600 students, how many were left-handed?

89. ***Area:*** A hexagonal dance floor has a total area of 720 square feet. If the floor separates into 6 triangles, each of which is $\frac{1}{6}$ the total area, find the area of one of the triangles.

90. ***Biking:*** If you go on a bicycle trip of 75 miles in the mountains and $\frac{1}{5}$ of the trip is downhill, what fraction of the trip is not downhill? How many miles are not downhill?

91. ***Sports:*** Suppose that a ball is dropped from a height of 40 feet and that each time the ball bounces it bounces back to $\frac{1}{2}$ the height it dropped. How high will the ball bounce on its third bounce?

92. ***Length:*** You have a wire that is 40 cm long. If you take $\frac{3}{4}$ of the wire and bend it into a square, find the length of one side of the square. You may want to draw a figure to help solve the problem. (**Hint:** $s = \frac{P}{4}$ where s is the side length of one side of the square and P is the perimeter of the square.)

93. ***Government:*** The student senate has 75 members, and $\frac{7}{15}$ of them are women. A change in the senate constitution is being considered, and at the present time (before debating has begun), a survey shows that $\frac{3}{5}$ of the women and $\frac{4}{5}$ of the men are in favor of this change.

a. How many women are on the student senate?

b. How many women on the senate are in favor of the change?

c. If the change requires $\frac{2}{3}$ majority vote in favor to pass, would the change pass if the vote were taken today?

d. By how many votes would the change pass or fail?

94. ***Education:*** There are 3000 students at Mountain High School, and $\frac{1}{4}$ of these students are seniors. If $\frac{3}{5}$ of the seniors are in favor of the school forming a debating team and $\frac{7}{10}$ of the remaining students (not seniors) are also in favor of forming a debating team, how many students do not favor this idea?

95. ***Politics:*** There are 4000 registered voters in Roseville and $\frac{3}{8}$ of these voters are registered Democrats. A survey indicates that $\frac{2}{3}$ of the registered Democrats are in favor of Bond Measure A and $\frac{3}{5}$ of the other registered voters are in favor of this measure.

a. How many of the voters are registered Democrats?

b. How many of the voters are not registered Democrats?

c. How many of the registered Democrats favor Measure A?

d. How many of the registered voters favor Measure A?

96. ***Education:*** A student can read $\frac{1}{5}$ of a book in 3 hours. If the book contains 450 pages, how many pages can he read in 3 hours?

Writing & Thinking

97. If two fractions are between 0 and 1, can their product be more than 1? Explain.

98. List the steps you would use in using prime factorization to reduce a fraction.

99. Explain the process of multiplying two fractions. Give an example of a product that cannot be reduced.

10.1 The Cartesian Coordinate System

Objectives

A. Relate ordered pairs of real numbers using an equation in two variables.

B. Plot ordered pairs of real numbers in the Cartesian coordinate system.

C. Find ordered pairs that satisfy a given linear equation.

D. Identify points on a graph.

René Descartes (1596–1650), a famous French mathematician, developed a system for solving geometric problems using algebra. This system is called the **Cartesian coordinate system** or the **rectangular coordinate system**. Descartes based his system on a relationship between points in a plane and **ordered pairs** of real numbers. This section begins by relating algebraic formulas with ordered pairs and then shows how these ideas can be related to geometry.

A Equations in Two Variables

The equation $d = 60t$ represents a relationship between the pair of variables t and d, where t is time and d is distance. For example, if a car is driven at 60 mph (the average speed) for $t = 3$ hours, then $d = 60 \cdot 3 = 180$ miles. With the understanding that t is first and d is second, we can represent $t = 3$ and $d = 180$ in the form of an **ordered pair** $(t, d) = (3, 180)$. We say that $(3, 180)$ **is a solution of** (or **satisfies**) the equation $d = 60t$. **The order of the numbers in an ordered pair is critical.** Thus, we see that the ordered pair $(3, 180)$ is different from $(180, 3)$.

In a similar way, the interest (I) earned on a principal (P) invested at 5% can be calculated with the equation $I = 0.05P$. Solutions are of the form (P, I) and one solution is $(100, 5)$, indicating that an investment of \$100 would earn \$5 in interest for one year.

For the equation $y = 2x + 3$, ordered pairs are in the form (x, y), and $(2, 7)$ satisfies the equation. If $x = 2$, then substituting in the equation gives $y = 2 \cdot 2 + 3 = 7$. In the ordered pair (x, y), x is called the **first coordinate** and y is called the **second coordinate**. To find ordered pairs that satisfy an equation in two variables, we can **choose any value** for one variable and find the corresponding value for the other variable by substituting into the equation. For example, **for the equation** $\boldsymbol{y = 2x + 3}$, we can find the following ordered pairs.

Choice for x	Substitution	Ordered Pair
$x = 1$	$y = 2(1) + 3 = 5$	$(1, 5)$
$x = -2$	$y = 2(-2) + 3 = -1$	$(-2, -1)$
$x = \frac{1}{2}$	$y = 2\left(\frac{1}{2}\right) + 3 = 4$	$\left(\frac{1}{2}, 4\right)$

Table 1

All the ordered pairs $(1, 5)$, $(-2, -1)$, and $\left(\frac{1}{2}, 4\right)$ satisfy the equation $y = 2x + 3$.

There are an infinite number of such ordered pairs. Any real number could have been chosen for x and the corresponding value for y calculated.

Since the equation $y = 2x + 3$ is solved for y, we say that the value of y "depends" on the choice of x. Thus, in an ordered pair of the form (x, y), the first coordinate x is called the **independent variable** and the second coordinate y is called the **dependent variable**.

In the following table, the first variable in each case is the independent variable and the second variable is the dependent variable. Corresponding ordered pairs would be of the form (t, d), (P, I), and (x, y). The choices for the values of the independent variables are arbitrary. There are an infinite number of other values that could have just as easily been chosen.

$d = 60t$			$I = 0.05P$			$y = 2x + 3$		
t	d	(t, d)	P	I	(P, I)	x	y	(x, y)
5	$60(5)$	$(5, 300)$	100	$0.05(100)$	$(100, 5)$	-2	$2(-2) + 3$	$(-2, -1)$
10	$60(10)$	$(10, 600)$	200	$0.05(200)$	$(200, 10)$	-1	$2(-1) + 3$	$(-1, 1)$
12	$60(12)$	$(12, 720)$	500	$0.05(500)$	$(500, 25)$	0	$2(0) + 3$	$(0, 3)$
15	$60(15)$	$(15, 900)$	1000	$0.05(1000)$	$(1000, 50)$	3	$2(3) + 3$	$(3, 9)$

Table 2

B Plotting Ordered Pairs

The Cartesian coordinate system relates algebraic equations and ordered pairs to geometry. In this system, two number lines intersect at right angles and separate the plane into four **quadrants**. The **origin**, designated by the ordered pair $(0, 0)$, is the point of intersection of the two lines. The horizontal number line is called the **horizontal axis** or ***x*-axis**. The vertical number line is called the **vertical axis** or ***y*-axis**. Points that lie on either axis are not in any quadrant. They are simply on an axis. (See Figure 1).

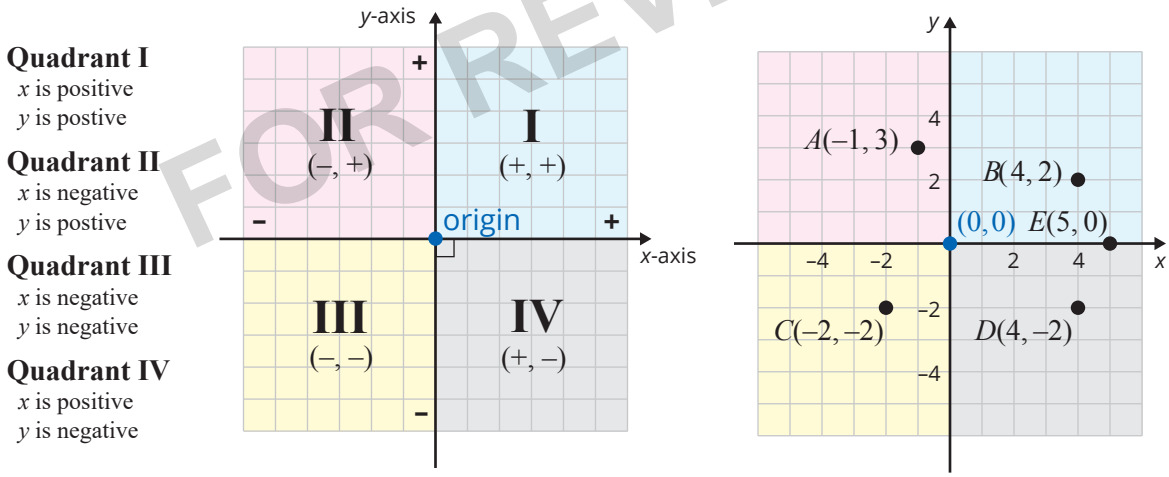


Figure 1

The following important relationship between ordered pairs of real numbers and points in a plane is the cornerstone of the Cartesian coordinate system.

One-to-One Correspondence

There is a **one-to-one correspondence** between points in a plane and ordered pairs of real numbers.

DEFINITION

In other words, for each point in a plane there is one and only one corresponding ordered pair of real numbers, and for each ordered pair of real numbers there is one and only one corresponding point in the plane.

The **points** $A(2,1), B(-2,3), C(-3,-2), D(1,-2)$, and $E(3,0)$ are shown on the graph in Figure 2. (**Note:** An ordered pair of real numbers and the corresponding point on the graph are frequently used to refer to each other. Thus, the ordered pair $(2,1)$ and the point $(2,1)$ are interchangeable ideas.)

Point	Quadrant
$A(2,1)$	I
$B(-2,3)$	II
$C(-3,-2)$	III
$D(1,-2)$	IV
$E(3,0)$	x-axis

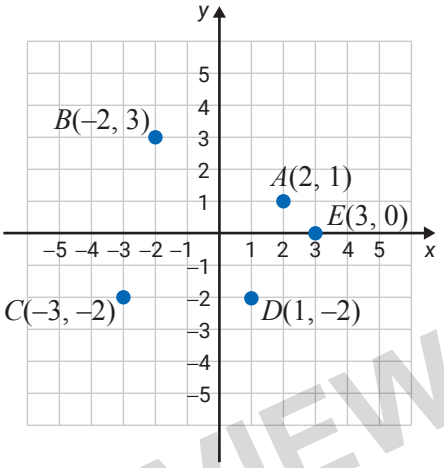


Figure 2

Notes

Although this discussion is related to ordered pairs of real numbers, most of the examples use ordered pairs of **integers**. This is because ordered pairs of integers are relatively easy to locate on and read from a graph. Ordered pairs with fractions, decimals, or radicals must be located by estimating the positions of the points. The precise coordinates intended for such points can be difficult or impossible to read because large dots must be used so the points can be seen. **Even with these difficulties, you should understand that we are discussing ordered pairs of real numbers and that points with fractions, decimals, and radicals as coordinates do exist and should be plotted by estimating their positions.**

Example 1 Plotting Ordered Pairs

Plot (or graph) the set of ordered pairs.

$$\{A(-2,1), B(-2,-4), C(0,4), D(1,3), E(2,-5)\}$$

Note: The listing of ordered pairs within the braces can be in any order.

1. Plot the set of ordered pairs, and label the points.

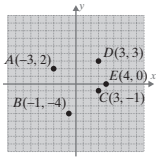
$\{A(-3,2), B(-1,-4), C(3,-1), D(3,3), E(4,0)\}$

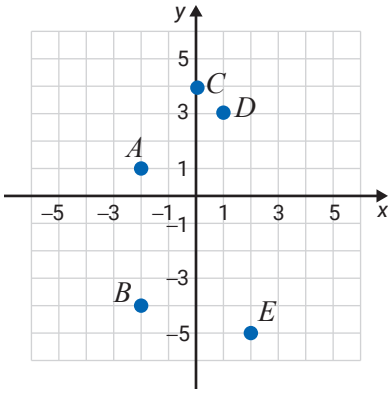
Solution

To plot each ordered pair, start at the **origin** $(0,0)$. For the x-coordinate, move right if positive and move left if negative. For the y-coordinate, move up if positive and move down if negative.

Answers

1.





For $A(-2, 1)$, move 2 units left and 1 unit up.

For $B(-2, -4)$, move 2 units left and 4 units down.

For $C(0, 4)$, move no units left or right and 4 units up.

For $D(1, 3)$, move 1 unit right and 3 units up.

For $E(2, -5)$, move 2 units right and 5 units down.

Now work margin exercise 1.

2. Plot the set of ordered pairs, and label the points.

$\{A(-3, -1), B(-2, 0), C(0, 2), D(3, 1), E(4, -3)\}$

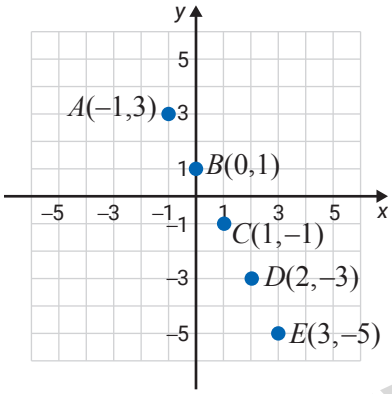
Example 2 Plotting Ordered Pairs

Plot (or graph) the set of ordered pairs.

$$\{A(-1, 3), B(0, 1), C(1, -1), D(2, -3), E(3, -5)\}$$

Solution

To plot each ordered pair, start at the **origin**, and move as follows.



For $A(-1, 3)$, move 1 unit left and 3 units up.

For $B(0, 1)$, move no units left or right and 1 unit up.

For $C(1, -1)$, move 1 unit right and 1 unit down.

For $D(2, -3)$, move 2 units right and 3 units down.

For $E(3, -5)$, move 3 units right and 5 units down.

Now work margin exercise 2.

C Finding Ordered Pairs that Satisfy Linear Equations

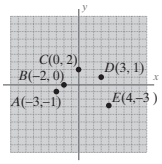
The points (ordered pairs) in Example 2 can be shown to satisfy the equation $y = -2x + 1$. For example, using $x = -1$ in the equation yields

$$\begin{aligned} y &= -2(-1) + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

and the ordered pair $(-1, 3)$ satisfies the equation. Similarly, letting $y = 1$ gives

Answers

2.



$$(1) = -2x + 1$$
$$0 = -2x$$
$$0 = x$$

and the ordered pair $(0, 1)$ also satisfies the equation.

We can write all the ordered pairs in Example 2 in table form.

x	$-2x + 1 = y$	(x,y)
-1	$-2(-1)+1=3$	$(-1, 3)$
0	$-2(0)+1=1$	$(0, 1)$
1	$-2(1)+1=-1$	$(1, -1)$
2	$-2(2)+1=-3$	$(2, -3)$
3	$-2(3)+1=-5$	$(3, -5)$

Table 3

Example 3 Finding Ordered Pairs

Find the missing coordinates in the ordered pairs so that each point will satisfy the equation $2x + 3y = 12$.

$$(0, \quad), (3, \quad), (\quad, 0), (\quad, -2)$$

Solution

The missing values can be found by substituting the given values for x (or for y) into the equation $2x + 3y = 12$ and solving for the other variable.

For $(0, \quad)$, let $x = 0$:

$$2(0) + 3y = 12$$
$$3y = 12$$
$$y = 4.$$

The ordered pair is $(0, 4)$.

For $(\quad, 0)$, let $y = 0$:

$$2x + 3(0) = 12$$
$$2x = 12$$
$$x = 6.$$

The ordered pair is $(6, 0)$.

For $(3, \quad)$, let $x = 3$:

$$2(3) + 3y = 12$$
$$6 + 3y = 12$$
$$3y = 6$$
$$y = 2.$$

The ordered pair is $(3, 2)$.

For $(\quad, -2)$, let $y = -2$:

$$2x + 3(-2) = 12$$
$$2x - 6 = 12$$
$$2x = 18$$
$$x = 9.$$

The ordered pair is $(9, -2)$.

Now work margin exercise 3.

3. Determine the missing coordinates in the ordered pairs so that each point will satisfy the equation $x + 5y = 15$.

$(0, \quad), (\quad, 2)$

$(15, \quad), (30, \quad)$

Answers

3. $(0, 3), (5, 2), (15, 0), (30, -3)$

4. Complete the table so that each ordered pair will satisfy the equation $y = -3x + 2$.

x	y	(x,y)
0		
	1	
-2		
	0	

Example 4 Finding Ordered Pairs

Complete the table so that each ordered pair will satisfy the equation $y = -3x + 1$.

x	y	(x, y)
0		
	4	
$\frac{1}{3}$		
3		

Solution

Substitut each given value for x or y into the equation $y = 1 - 2x$ to find the ordered pairs and complete the table.

For $x = 0$:

$y = -3(0) + 1$

$y = 0 + 1$

$y = 1$

For $y = 4$:

$(4) = -3x + 1$

$3 = -3x$

$-1 = x$

For $x = \frac{1}{3}$:

$y = -3\left(\frac{1}{3}\right) + 1$

$y = -1 + 1$

$y = 0$

For $x = 3$:

$y = -3(3) + 1$

$y = -9 + 1$

$y = -8$

The completed table is as follows.

x	y	(x, y)
0	1	$(0, 1)$
-1	4	$(-1, 4)$
$\frac{1}{3}$	0	$\left(\frac{1}{3}, 0\right)$
3	-8	$(3, -8)$

Now work margin exercise 4.

5. Determine which, if any, of the ordered pairs $(0, -3)$, $(3, 6)$, and $\left(\frac{1}{4}, -2\right)$ satisfy the equation $y = 4x - 3$.

Example 5 Determining Ordered Pairs

Determine which, if any, of the ordered pairs $(0, -2)$, $\left(\frac{2}{3}, 0\right)$, and $(2, 5)$ satisfy the equation $y = 3x - 2$.

Solution

To determine whether an ordered pair is a solution to an equation, substitute the x-coordinate for x and the y-coordinate for y in the equation $y = 3x - 2$ to see if the result is a true statement.

For the ordered pair $(0, -2)$, let $x = 0$ and $y = -2$.

$(-2) \stackrel{?}{=} 3(0) - 2$

$-2 \stackrel{?}{=} 0 - 2$

$-2 = -2$ true statement

Answers

4.

x	y	(x, y)
0	2	$(0, 2)$
$\frac{1}{3}$	1	$\left(\frac{1}{3}, 1\right)$
-2	8	$(-2, 8)$
$\frac{2}{3}$	0	$\left(\frac{2}{3}, 0\right)$

5. $(0, -3)$ and $\left(\frac{1}{4}, -2\right)$ satisfy the equation $y = 4x - 3$.

For the ordered pair $\left(\frac{2}{3}, 0\right)$, let $x = \frac{2}{3}$ and $y = 0$.

$$(0) \stackrel{?}{=} 3\left(\frac{2}{3}\right) - 2$$

$$0 \stackrel{?}{=} 2 - 2$$

$$0 = 0 \qquad \text{true statement}$$

For the ordered pair $(2, 5)$, let $x = 2$ and $y = 5$.

$$(5) \stackrel{?}{=} 3(2) - 2$$

$$5 \stackrel{?}{=} 6 - 2$$

$$5 \neq 4 \qquad \text{false statement}$$

So the ordered pairs $(0, -2)$ and $\left(\frac{2}{3}, 0\right)$ satisfy the equation, and the ordered pair $(2, 5)$ does not.

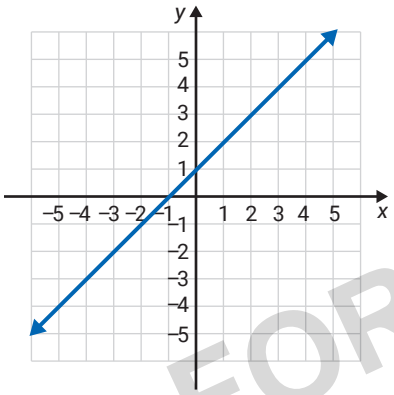
Now work margin exercise 5.

D Identifying Points on a Graph

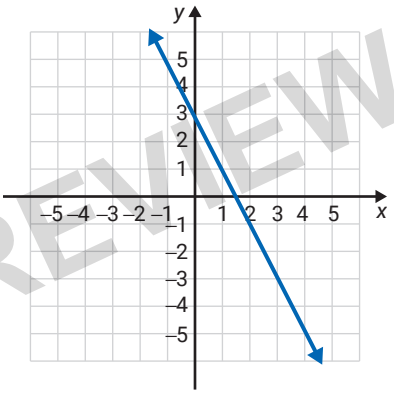
Example 6 Locating Points on the Graph of a Line

The graphs of two lines are given. Each line contains an infinite number of points. Use the grid to help you locate (or estimate) three points on each line.

a.



b.



Solution

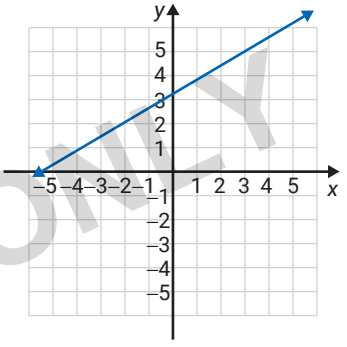
a. Three points on this graph are $(-2, -1)$, $(1, 2)$, and $(3, 4)$. (Of course there is more than one correct answer to this type of question. Use your own judgment.)

b. Three points on this graph are $(0, 3)$, $(1, 1)$ and $(2, -1)$. (You may also estimate with fractions. For example, one point appears to be approximately $\left(\frac{1}{2}, 2\right)$.)

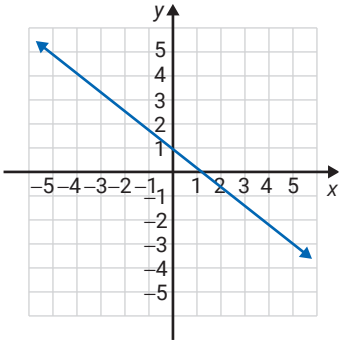
Now work margin exercise 6.

6. The graphs of two lines are given. Use the grid to locate three points on each line.

a.



b.



Answers

6. a. $(-5, -2)$, $(0, 1)$, $(5, 4)$

b. $(-5, 5)$, $(0, 1)$, $(5, -3)$

10.1 Exercises

Concept Check

Fill-in-the-Blank. Complete the sentences using information found in this chapter.

1. The Cartesian coordinate system has a vertical and a horizontal line that separate a plane into four _____.
2. In an ordered pair, x represents the _____ (first/second) coordinate and y represents the _____ (first/second) coordinate.
3. If an ordered pair has two negative coordinates, the graph of the corresponding point is in Quadrant _____.
4. If an ordered pair satisfies an equation, it is a/an _____ of the equation.
5. The point of intersection of the x-axis and y-axis is called the _____.
6. Linear equations have a/an _____ number of solutions.

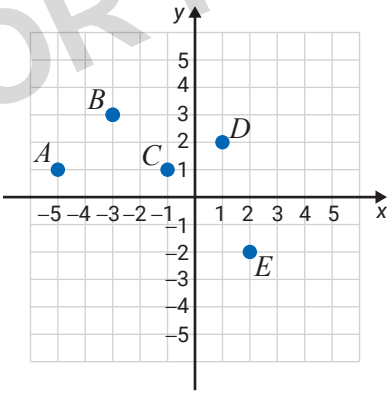
True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

7. The graph of every ordered pair that has a positive x-coordinate and a negative y-coordinate can be found in Quadrant IV.
8. To find the y-value that corresponds with $x = 2$, substitute 2 for x into the given equation and solve for y.
9. If $(-7, 3)$ is a solution of $y = 3x + 24$, then $(-7, 3)$ satisfies $y = 3x + 24$.
10. If point $A = (0, 4)$, then point A lies on the x-axis.

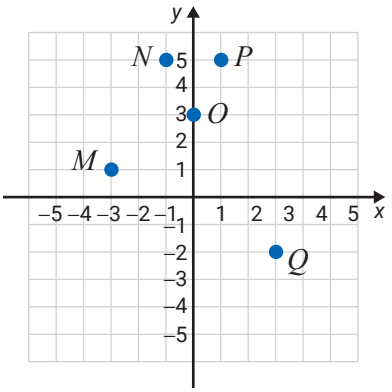
Practice

For each graph, list the set of ordered pairs corresponding to the points on the graph.

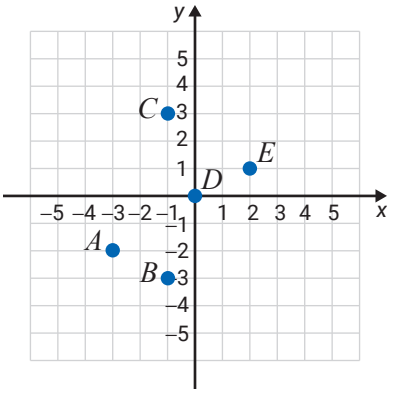
1.



2.



3.



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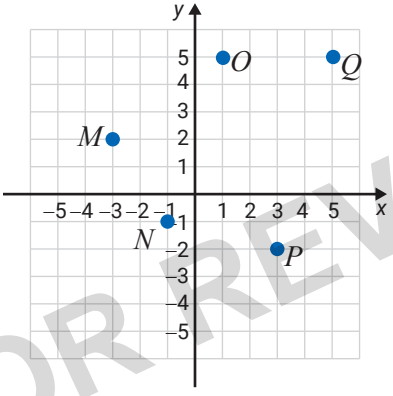
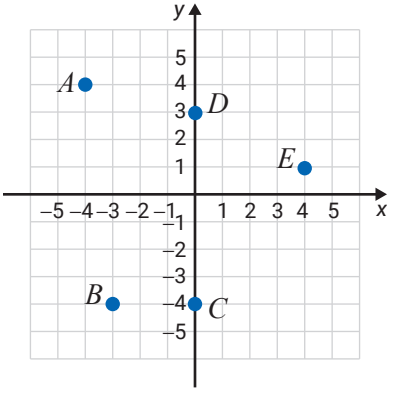
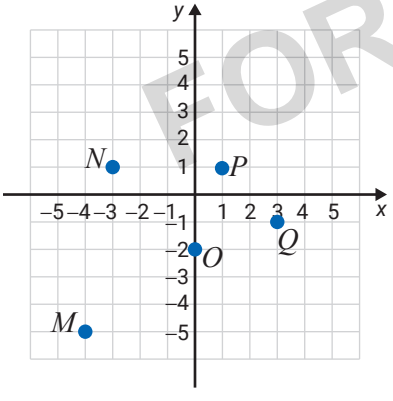


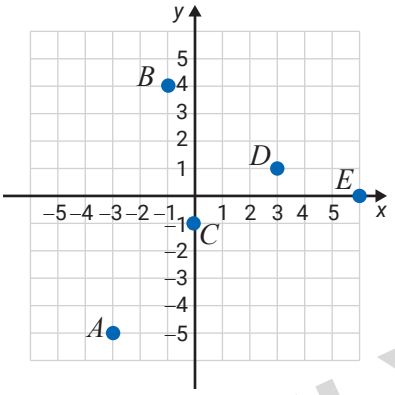
5.



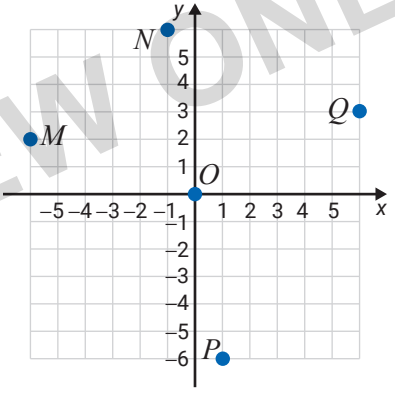
6.



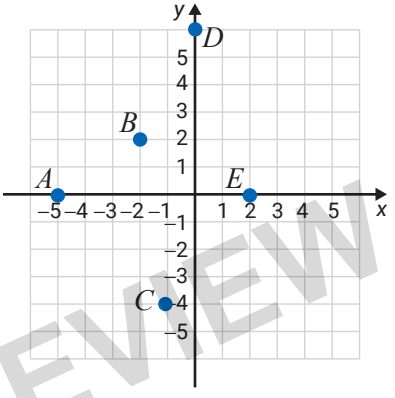
7.



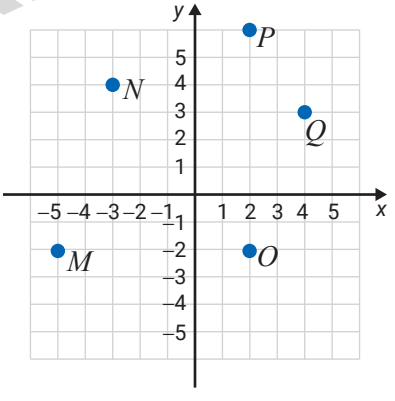
8.



9.



10.



Plot each set of ordered pairs and label the points. See Examples 1 and 2.

11. $\{A(4,-1), B(3,2), C(0,5), D(1,-1), E(1,4)\}$

12. $\{A(-1,-1), B(-3,-2), C(1,3), D(0,0), E(2,5)\}$

13. $\{A(1,2), B(0,2), C(-1,2), D(2,2), E(-3,2)\}$

14. $\{A(-1,4), B(0,-3), C(2,-1), D(4,1), E(-1,-1)\}$

15. $\{A(1,0), B(3,0), C(-2,1), D(-1,1), E(0,0)\}$

16. $\{A(-1,-1), B(0,1), C(1,3), D(2,5), E(3,10)\}$

17. $\{A(4,1), B(0,-3), C(1,-2), D(2,-1), E(-4,2)\}$

18. $\{A(0,1), B(1,0), C(2,-1), D(3,-2), E(4,-3)\}$

19. $\{A(1,4), B(-1,-2), C(0,1), D(2,7), E(-2,-5)\}$

20. $\{A(0,0), B(-1,3), C(3,-2), D(0,4), E(-7,0)\}$

21. $\left\{A(1,-3), B\left(-4,\frac{3}{4}\right), C\left(2,-2\frac{1}{2}\right), D\left(\frac{1}{2},4\right)\right\}$

22. $\left\{A\left(\frac{3}{4},\frac{1}{2}\right), B\left(2,-\frac{5}{4}\right), C\left(\frac{1}{3},-2\right), D\left(-\frac{5}{3},2\right)\right\}$

23. $\{A(1.6,-2), B(3,2.5), C(-1,1.5), D(0,-2.3)\}$

24. $\{A(-2,2), B(-3,1.6), C(3,0.5), D(1.4,0)\}$

Determine which, if any, of the ordered pairs satisfy the given equations. See Example 2.

25. $2x - y = 4$

a. $(1,1)$
b. $(2,0)$
c. $(1,-2)$
d. $(3,2)$

26. $x + 2y = -1$

a. $(1,-1)$
b. $(1,0)$
c. $(2,1)$
d. $(3,-2)$

27. $4x + y = 5$

a. $\left(\frac{3}{4},2\right)$
b. $(4,0)$
c. $(1,1)$
d. $(0,3)$

28. $2x - 3y = 7$

a. $(1,3)$
b. $\left(\frac{1}{2},-2\right)$
c. $\left(\frac{7}{2},0\right)$
d. $(2,1)$

29. $2x + 5y = 8$

a. $(4,0)$
b. $(2,1)$
c. $(1,1.2)$
d. $(1.5,1)$

30. $3x + 4y = 10$

a. $(-2,3)$
b. $(0,2.5)$
c. $(4,-2)$
d. $(1.2,1.6)$

Determine the missing coordinate in each of the ordered pairs so that the point will satisfy the equation given. See Example 3.

31. $x - y = 4$

a. $(0, \_\_)$

b. $(2, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, -3)$

32. $x + y = 7$

a. $(0, \_\_)$

b. $(-1, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 3)$

33. $x + 2y = 6$

a. $(0, \_\_)$

b. $(2, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 4)$

34. $3x + y = 9$

a. $(0, \_\_)$

b. $(4, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 3)$

35. $4x - y = 8$

a. $(0, \_\_)$

b. $(1, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 4)$

36. $x - 2y = 2$

a. $(0, \_\_)$

b. $(4, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 3)$

37. $2x + 3y = 6$

a. $(0, \_\_)$

b. $(-1, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, -2)$

38. $5x + 3y = 15$

a. $(0, \_\_)$

b. $(2, \_\_)$

c. $(\_\_, 0)$

d. $(\_\_, 4)$

39. $3x - 4y = 7$

a. $(0, \_\_)$

b. $(1, \_\_)$

c. $(\_\_, 0)$

d. $\left(\_\_, \frac{1}{2}\right)$

40. $2x + 5y = 6$

a. $(0, \_\_)$

b. $\left(\frac{1}{2}, \_\_\right)$

c. $(\_\_, 0)$

d. $(\_\_, 2)$

Complete the tables so that each ordered pair will satisfy the given equation. Plot the resulting sets of ordered pairs. See Example 4.

41. $y = 3x$

x	y
0	
	−3
−2	
	6

42. $y = -2x$

x	y
0	
	4
3	
	−2

43. $y = 2x - 3$

x	y
0	
	−1
−2	
	3

44. $y = 3x + 5$

x	y
0	
	−4
−2	
	2

45. $y = 9 - 3x$

x	y
0	
	0
1	
	−3

46. $y = 6 - 2x$

x	y
0	
	0
−2	
	−2

47. $y = \frac{3}{4}x + 2$

x	y
0	
	5
−4	
	$\frac{5}{4}$

48. $y = \frac{3}{2}x - 1$

x	y
0	
	2
−2	
	$-\frac{5}{2}$

49. $3x - 5y = 9$

x	y
0	
	0
−2	
	−1

50. $4x + 3y = 6$

x	y
0	
	0
3	
	−1

51. $5x - 2y = 10$

x	y
0	
	0
−1	
	5

52. $3x - 2y = 12$

x	y
	0
0	
	−3
6	

53. $2x + 3.2y = 6.4$

x	y
0	
3.2	
	0.8
	−0.2

54. $3x + y = -2.4$

x	y
	0
0	
	0.6
1.6	

Determine which, if any, of the ordered pairs satisfy the given equations. See Example 5.

55. $y = 2x - 4$

a. $(1, 1)$

b. $(2, 0)$

c. $(1, -2)$

d. $(3, 2)$

56. $y = -4x + 5$

a. $\left(\frac{3}{4}, 2\right)$

b. $(4, 0)$

c. $(1, 1)$

d. $(0, 3)$

57. $x + 2y = -1$

a. $(1, -1)$

b. $(1, 0)$

c. $(2, 1)$

d. $(3, -2)$

58. $2x - 3y = 7$

a. $(1, 3)$

b. $\left(\frac{1}{2}, -2\right)$

c. $\left(\frac{7}{2}, 0\right)$

d. $(2, 1)$

59. $2x + 5y = 8$

a. $(4, 0)$

b. $(2, 1)$

c. $(1, 1.2)$

d. $(1.5, 1)$

60. $3x + 4y = 10$

a. $(-2, 3)$

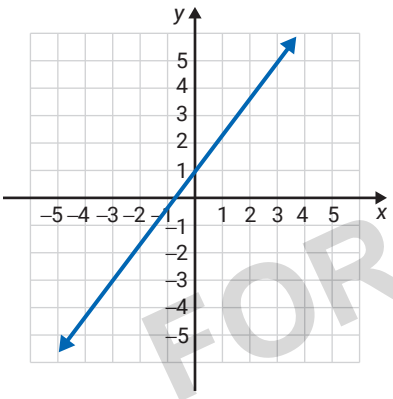
b. $(0, 2.5)$

c. $(4, -2)$

d. $(1.2, 1.6)$

The graph of a line is shown. List any three points on each line. (There is more than one correct answer.) See Example 6.

61.



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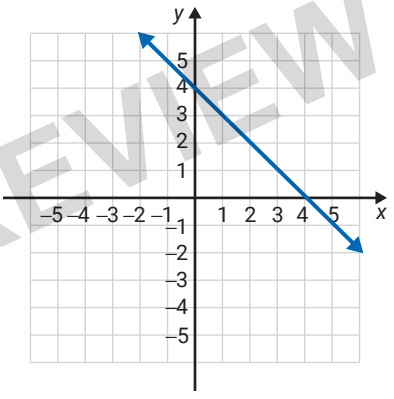

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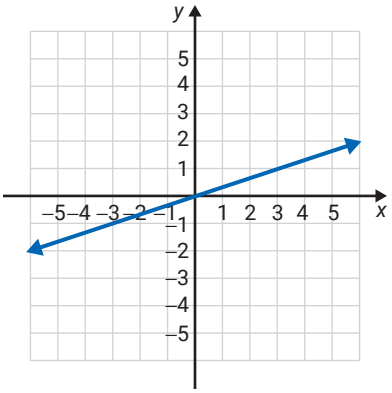

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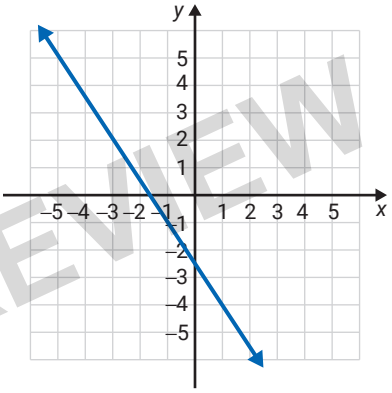

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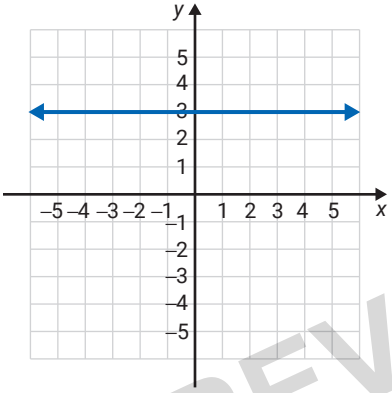

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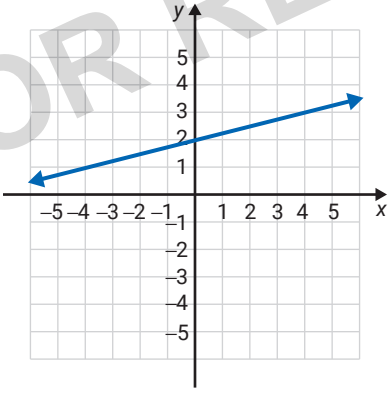

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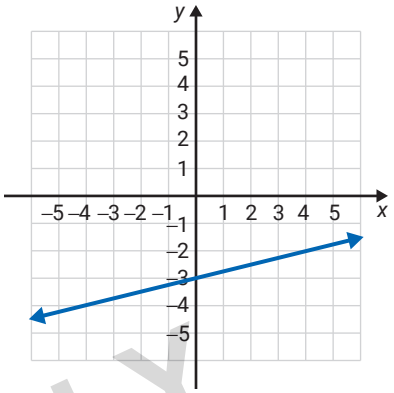

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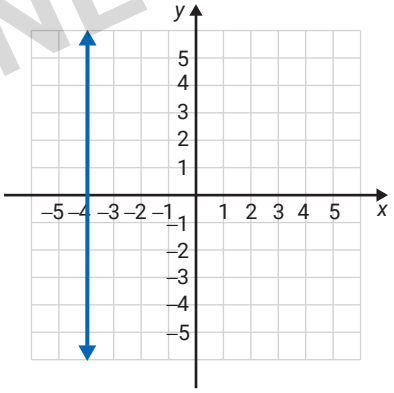

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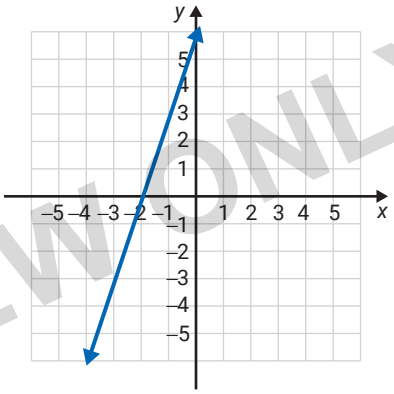

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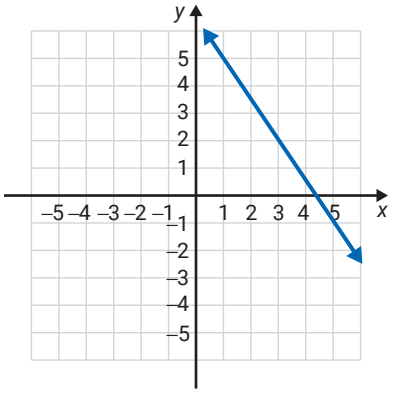


Applications

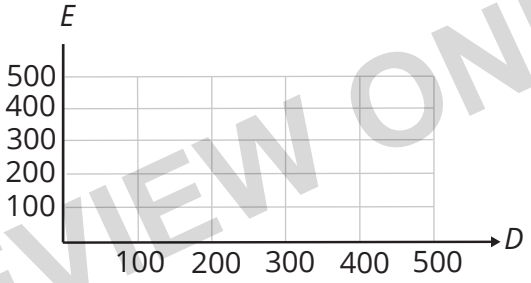
Solve.

71. ***Exchange Rates:*** At one point in 2017, the exchange rate from US dollars to Euros was $E = 0.85D$ where E is Euros and D is dollars.

a. Make a table of ordered pairs for the values of D and E if D has the values \$100, \$200, \$300, \$400, and \$500.

b. Plot the points corresponding to the ordered pairs.

D	E
100	
200	
300	
400	
500	

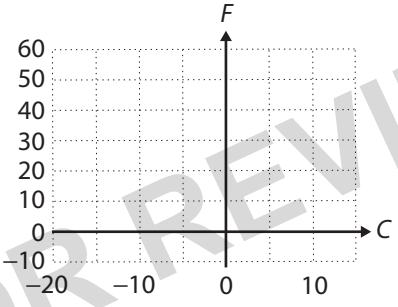


72. ***Temperature:*** Given the equation $F = \frac{9}{5}C + 32$ where C is temperature in degrees Celsius and F is the corresponding temperature in degrees Fahrenheit:

a. Make a table of ordered pairs for the values of C and F if C has the values –20°, –10°, –5°, 0°, 5°, 10°, and 15°.

b. Plot the points corresponding to the ordered pairs.

C	F
–20	
–10	
–5	
0	
5	
10	
15	

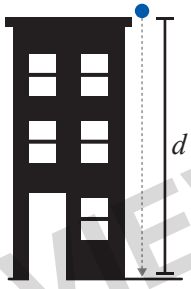


73. ***Falling Objects:*** Given the equation $d = 16t^2$, where d is the distance an object falls in feet and t is the time in seconds that the object falls:

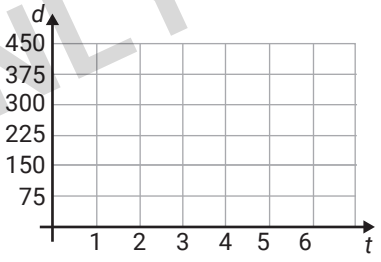
a. Make a table of ordered pairs for the values of t and d with the values of 1, 2, 3.5, 4, 4.5, and 5 seconds for t.

b. Plot the points corresponding to the ordered pairs.

c. These points do not lie on a straight line. What feature of the equation might indicate to you that the graph is not a straight line?



t	d
1	
2	
3.5	
4	
4.5	
5	

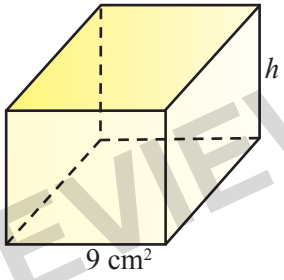


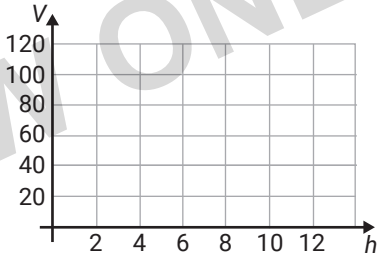
74. ***Volume:*** Given the equation $V = 9h$, where V is the volume (in cubic centimeters) of a box with a variable height h in centimeters and a fixed base of area 9 cm^2.

a. Make a table of ordered pairs for the values of h and V with h as the values 2 cm, 3 cm, 5 cm, 8 cm, 9 cm, and 10 cm.

b. Plot the points corresponding to the ordered pairs.

h	V
2	
3	
5	
8	
9	
10	





75. ***Sales:*** A business owner records the number of customers per hour to determine peak shopping times after noon. Graph the points corresponding to the ordered pairs.

Hour of the Day	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.	8 p.m.
Number of Customers	500	450	200	650	900	700	550	300

Writing & Thinking

In statistics, data is sometimes given in the form of ordered pairs where each ordered pair represents two pieces of information about one person. For example, ordered pairs might represent the height and weight of a person or the person's number of years of education and that person's annual income. The ordered pairs are plotted on a graph and the graph is called a **scatter diagram (or scatter plot)**. Such scatter diagrams are used to see if there is any pattern to the data and, if there is, then the diagram is used to predict the value for one of the variables if the value of the other is known. For example, if you know that a person's height is 5 ft 6 in., then his or her weight might be predicted from information indicated in a scatter diagram that has several points of known information about height and weight.

Solve.

76. a. The following table of values indicates the number of push-ups and the number of sit-ups that ten students did in a physical education class. Plot these points in a scatter diagram.

Person	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
x(push-ups)	20	15	25	23	35	30	42	40	25	35
y(sit-ups)	25	20	20	30	32	36	40	45	18	40

b. Does there seem to be a pattern in the relationship between push-ups and sit-ups? What is this pattern?

c. Using the scatter diagram in part **a.**, predict the number of sit-ups that a student might be able to do if he or she has just done each of the following numbers of push-ups: 22, 32, 35, and 45. (**Note:** In each case, there is no one correct answer. The answers are only estimates based on the diagram.)

77. Ask ten friends or fellow students what their heights and shoe sizes are. (You may want to ask all men or all women since the scales for men's and women's shoe sizes are different.) Organize the data in table form and then plot the corresponding scatter diagram. Knowing your own height, does the pattern indicated in the scatter diagram seem to predict your shoe size?

78. Ask ten friends or fellow students what their heights and ages are. Organize the data in table form and then plot the corresponding scatter diagram. Knowing your own height, does the pattern indicated in the scatter diagram seem to predict your age? Do you think that all scatter diagrams can be used to predict information related to the two variables graphed? Explain.

Objectives

A. Graph linear equations by plotting points on the line.

B. Graph linear equations by finding and plotting their x- and y-intercepts.

C. Graph horizontal and vertical lines.

10.2 Graphing Linear Equations in Two Variables

A Graphing Linear Equations by Plotting Points

In Section 10.1, we discussed ordered pairs of real numbers and graphed a few points (ordered pairs) that satisfied particular equations. Now, suppose we want to graph all the points that satisfy an equation such as

$$y = -3x + 3.$$

The **solution set** for equations of this type (in the two variables x and y) consists of an infinite set of ordered pairs in the form (x, y) that satisfy the equation.

To find some of the solutions of the equation $y = -3x + 3$, we form a table (as we did in Section 10.1) by

1. choosing arbitrary values for x and
2. finding the corresponding values for y by substituting into the equation.

In Figure 1, we have found five ordered pairs that satisfy the equation and graphed the corresponding points.

Choices x	Substitutions $-3x + 3 = y$	Results (x,y)
-1	$-3(-1)+3=6$	$(-1, 6)$
0	$-3(0)+3=3$	$(0, 3)$
$\frac{2}{3}$	$-3\left(\frac{2}{3}\right)+3=1$	$\left(\frac{2}{3}, 1\right)$
2	$-3(2)+3=-3$	$(2, -3)$
3	$-3(3)+3=-6$	$(3, -6)$

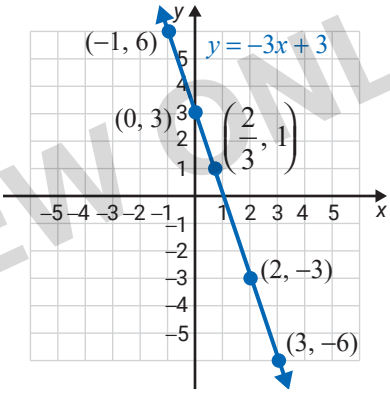


Figure 1

The five points in Figure 1 appear to lie on a line. They do in fact lie on a line, and any ordered pair that satisfies the equation $y = -3x + 3$ will also lie on that same line.

Just as we use the terms **ordered pair** and **point** (the graph of an ordered pair) interchangeably, we use the terms **equation** and **graph of an equation** interchangeably. The equations

$$2x + 3y = 4, \quad y = -5, \quad x = 1.4, \quad \text{and} \quad y = 3x + 2$$

are called **linear equations**, and their graphs are lines on the Cartesian plane.

Standard Form of a Linear Equation

Any equation of the form

$$Ax + By = C,$$

where A, B, and C are real numbers and A and B are not both equal to 0, is called the **standard form of a linear equation**.

DEFINITION

Note

Note that in the standard form $Ax + By = C$, A and B may be positive, negative, or 0, but A and B cannot **both** equal 0.

Every line corresponds to some linear equation, and the graph of every linear equation is a line. We know from geometry that **two points determine a line**. This means that the graph of a linear equation can be found by locating any two points that satisfy the equation.

To Graph a Linear Equation in Two Variables

1. Locate any two points that satisfy the equation. (Choose values for x and y that lead to values that are easily calculated for the other variable. Remember that there are an infinite number of choices for either x or y. Once a value for x or y is chosen, the corresponding value for the other variable is found by substituting into the equation.)
2. Plot these two points on a Cartesian coordinate system.
3. Draw a line through these two points. (**Note:** Every point on that line will satisfy the equation.)
4. **To check:** Locate a third point that satisfies the equation and check to see that it does indeed lie on the line.

PROCEDURE

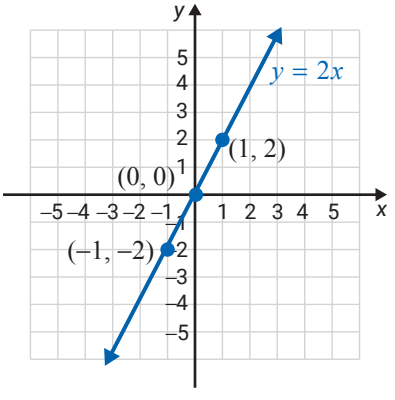
Example 1 Graphing a Linear Equation in Two Variables

Graph: $y = 2x$

Solution

Substitute −1, 0, and 1 for x.

x	$y = 2x$	y
−1	$y = 2(-1)$	−2
0	$y = 2(0)$	0
1	$y = 2(1)$	2

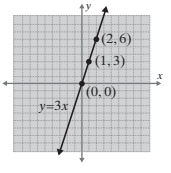


Now work margin exercise 1.

1. Graph: $y = 3x$

Answers

1.



2. Graph: $3x + 2y = 6$

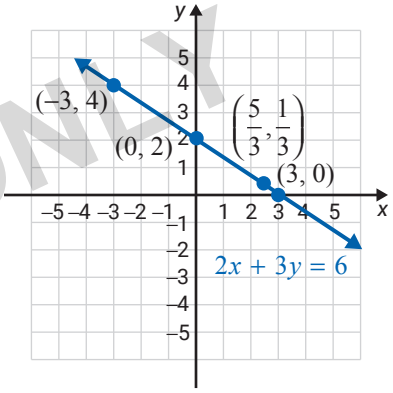
Example 2 Graphing a Linear Equation in Two Variables

Graph: $2x + 3y = 6$

Solution

Make a table with headings x and y and, whenever possible, **choose values for x or y that lead to values that are easily calculated for the other variable.** (Values chosen for x and y are colored and bolded.)

x	$2x + 3y = 6$	y
0	$2(0) + 3y = 6$	2
−3	$2(-3) + 3y = 6$	4
3	$2x + 3(0) = 6$	**0**
$\frac{5}{2}$	$2x + 3\left(\frac{1}{3}\right) = 6$	$\frac{1}{3}$



Now work margin exercise 2.

3. Graph: $2x - y = 2$

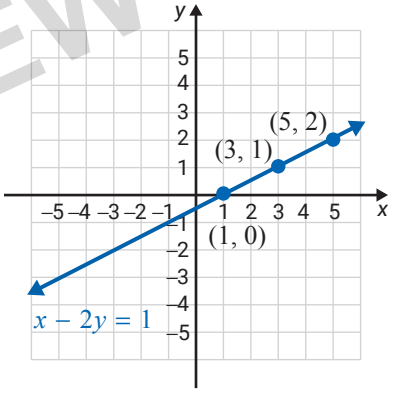
Example 3 Graphing a Linear Equation in Two Variables

Graph: $x - 2y = 1$

Solution

Solve the equation for x ($x = 2y + 1$) and substitute 0, 1, and 2 for y.

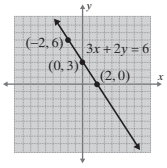
x	$x = 2y + 1$	y
1	$x = 2(0) + 1$	**0**
3	$x = 2(1) + 1$	**1**
5	$x = 2(2) + 1$	**2**



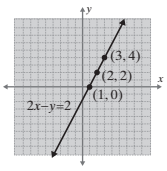
Now work margin exercise 3.

Answers

2.



3.

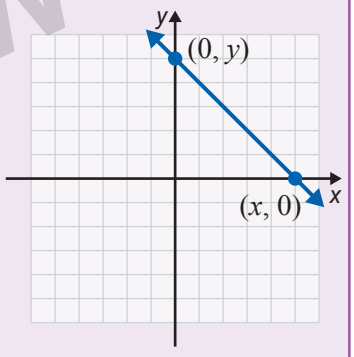


B Using *x*- and *y*-Intercepts to Graph Linear Equations

While the choice of the values for x or y can be arbitrary, letting $x = 0$ will locate the point on the graph where the line crosses (or intercepts) the y-axis. This point is called the ***y*-intercept** and is of the form $(0, y)$. The ***x*-intercept** is the point found by letting $y = 0$. This is the point where the line crosses (or intercepts) the x-axis and is of the form $(x, 0)$. These two points are generally easy to locate and are frequently used as the two points for drawing the graph of a linear equation. If a line passes through the point $(0, 0)$, then the y-intercept and the x-intercept are the same point; namely, the origin (see Example 1). In this case, you will also need to locate some other point to draw the graph.

Intercepts

1. To find the ***y*-intercept** (where the line crosses the y-axis), substitute $x = 0$ and solve for y.
2. To find the ***x*-intercept** (where the line crosses the x-axis), substitute $y = 0$ and solve for x.



PROCEDURE

Note

In general, the intercepts are easy to find because substituting 0 for x or y leads to an easy solution for the other variable. However, when the intercepts result in a point with fractional (or decimal) coordinates and estimation is involved, then a third point that satisfies the equation should be found to verify that the line is graphed correctly.

Example 4 Using Intercepts to Graph Linear Equations

Graph $x + 3y = 9$ by locating the y-intercept and the x-intercept.

Solution

Find the y-intercept:

$$x = 0 \rightarrow (0) + 3y = 9$$
$$3y = 9$$
$$y = 3$$

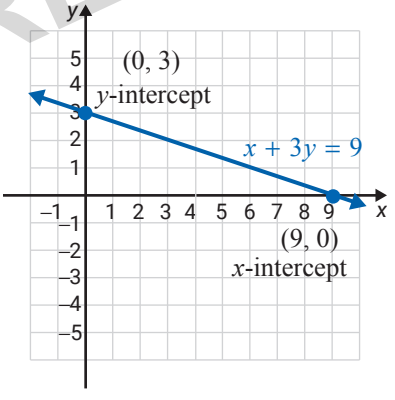
$(0, 3)$ is the y-intercept.

Find the x-intercept:

$$y = 0 \rightarrow x + 3(0) = 9$$
$$x = 9$$

$(9, 0)$ is the x-intercept.

Plot the two intercepts and draw the line that contains them.

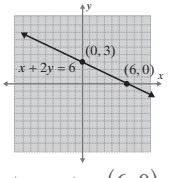


Now work margin exercise 4.

4. Graph $x + 2y = 6$ by locating the x-intercept and the y-intercept.

Answers

4.



x-intercept $= (6, 0)$

y-intercept $= (0, 3)$

5. Graph $5x - 3y = 15$ by locating the x-intercept and the y-intercept.

Example 5 Using Intercepts to Graph Linear Equations

Graph $3x - 2y = 12$ by locating the y-intercept and the x-intercept.

Solution

Find the y-intercept:

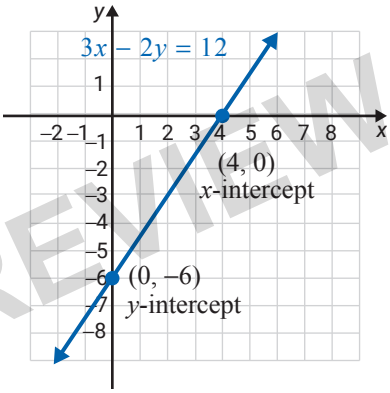
$$x = 0 \rightarrow 3(0) - 2y = 12$$
$$-2y = 12$$
$$y = -6$$

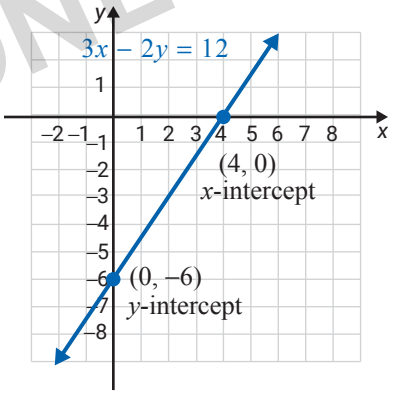
$(0, -6)$ is the y-intercept.

Find the x-intercept:

$$y = 0 \rightarrow 3x - 2(0) = 12$$
$$3x = 12$$
$$x = 4$$

$(4, 0)$ is the x-intercept.





Plot the two intercepts and draw the line that contains them.

Now work margin exercise 5.

6. Graph $4x - y = 8$ by locating the x-intercept and the y-intercept.

Completion Example 6 Using Intercepts to Graph Equations

Graph $x - 5y = 5$ by locating the y-intercept and the x-intercept.

Solution

Find the y-intercept:

$x = 0 \rightarrow$ ___ $- 5y = 5$

_____

_____

_____ is the y-intercept.

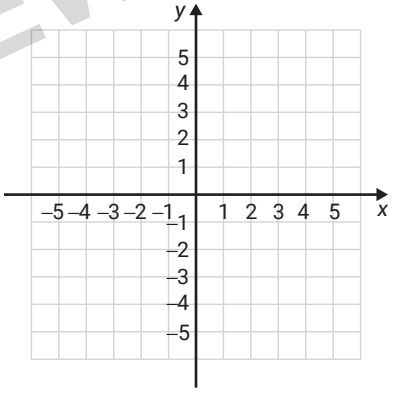
Find the x-intercept:

$y = 0 \rightarrow x - 5 \cdot$ ___ $= 5$

_____

_____

_____ is the x-intercept.

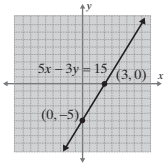


Plot the two intercepts and draw the line that contains them.

Now work margin exercise 6.

Answers

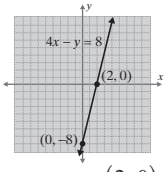
5.



x-intercept = $(3, 0)$

y-intercept = $(0, -5)$

6.

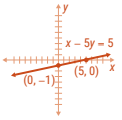


x-intercept = $(2, 0)$

y-intercept = $(0, -8)$

Completion Example Answer

6. $x = 0 \rightarrow 0 - 5y = 5$; $-5y = 5$; $y = -1$; $(0, -1)$ is the y-intercept;
$y = 0 \rightarrow x - 5 \cdot 0 = 5$; $x - 0 = 5$; $x = 5$; $(5, 0)$ is the x-intercept.



C Graphing Horizontal and Vertical Lines

Now consider the linear equation $y = 4$ (which can be written in standard form as $0x + y = 4$). Regardless of the value substituted for x, the corresponding y-value will be 4. The graph of this equation is a **horizontal line** with y-intercept $(0, 4)$, as illustrated in Example 7.

Example 7 Graphing Horizontal Lines

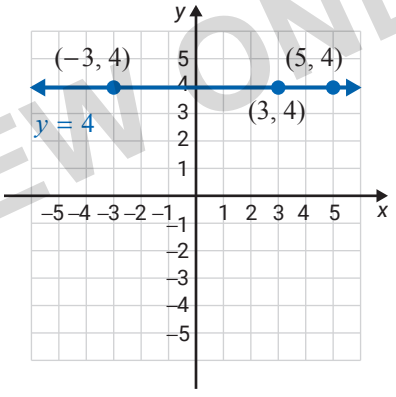
7. Graph the line $y = -1$

Graph the line $y = 4$ (or $0x + y = 4$).

Solution

Choose three values for x; for example, –3, 3, and 5. As indicated in the following table, $y = 4$ in each case. The graph is a horizontal line.

x	$0x + y = 4$	y
–3	$0(-3) + y = 4$	4
3	$0(3) + y = 4$	4
5	$0(5) + y = 4$	4



Now work margin exercise 7.

Next, consider the linear equation $x = -2$ (which can be written in standard form as $x + 0y = -2$). Regardless of the value substituted for y, the corresponding x-value will be –2. The graph of this equation is a **vertical line** with x-intercept $(-2, 0)$ as illustrated in Example 8.

Example 8 Graphing Vertical Lines

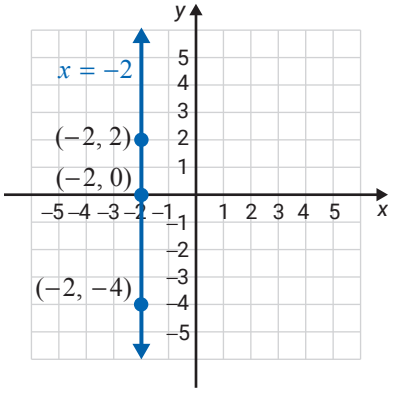
8. Graph the line $x = 4$

Graph the line $x = -2$ (or $x + 0y = -2$).

Solution

Choose three values for y; for example −4, 0, and 2. As indicated in the following table, $x = -2$ in each case. The graph is a vertical line.

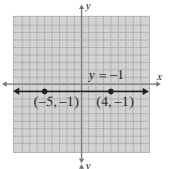
x	$0x + y = 4$	y
−2	$x + 0(-4) = -2$	−4
−2	$x + 0(0) = -2$	0
−2	$x + 0(2) = -2$	2



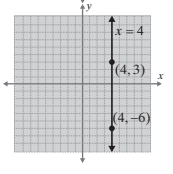
Now work margin exercise 8.

Answers

7.



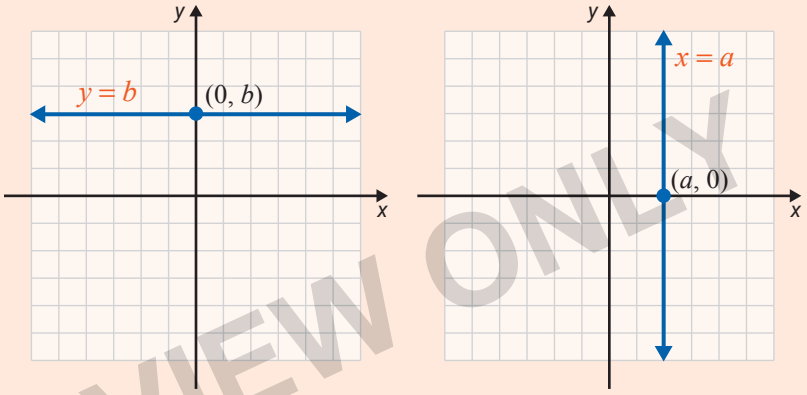
8.



Horizontal and Vertical Lines

For real numbers a and b, the graph of

$y = b$ is a **horizontal line** and $x = a$ is a **vertical line.**



DEFINITION

10.2 Exercises

Concept Check

Fill-in-the-Blank. Complete the sentences using information found in this chapter.

1. If 0 is substituted for x in a linear equation and the resulting equation is solved for y, the result will be the ___-intercept.
2. If 0 is substituted for y in a linear equation and the resulting equation is solved for x, the result will be the ___-intercept.
3. The solution set for linear equations is a/an _____ set of ordered pairs.
4. The standard form of a linear equation is _____.
5. The graph of every linear equation is a/an _____.
6. The graph of a line is determined by ___ points.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

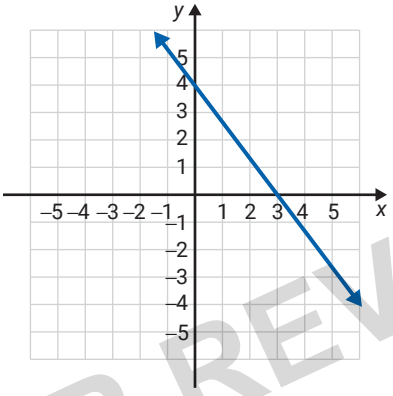
7. The y-intercept is the point where a line crosses the y-axis.
8. The terms ordered pair and point are used interchangeably.
9. A horizontal line does not have a y-intercept.
10. All x-intercepts correspond to an ordered pair of the form $(0, y)$.

Practice

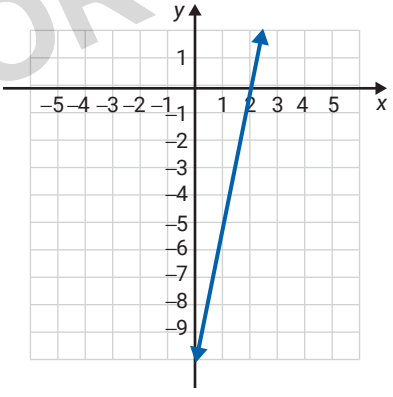
Use your knowledge of *y*-intercepts and *x*-intercepts to match each of the following equations with its graph.

1. $4x + 3y = 12$
2. $4x - 3y = 12$
3. $x + 2y = 8$
4. $-x + 2y = 8$
5. $x + 4y = 0$
6. $5x - y = 10$

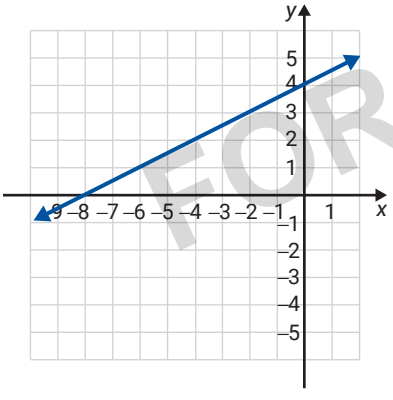
a.



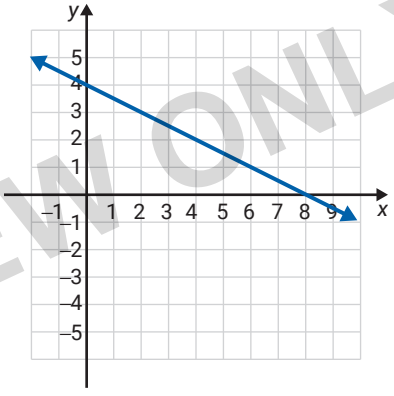
b.



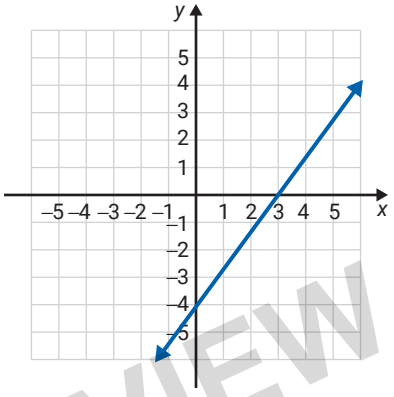
c.



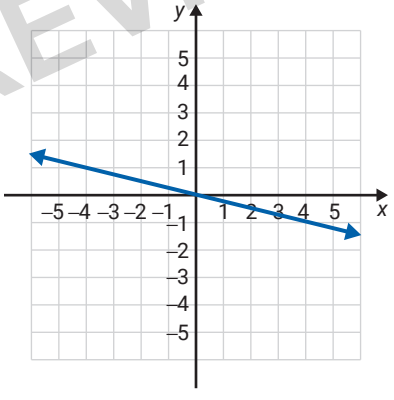
d.



e.



f.



Graph each linear equation by locating at least two ordered pairs that satisfy the given equation. See Examples 1 through 3.

7. $x + y = 3$

8. $x + y = 4$

9. $y = x$

10. $2y = x$

11. $2x + y = 0$

12. $x = 1$

13. $3x + 2y = 0$

14. $2x + 3y = 7$

15. $x + 2 = 0$

16. $4x + 3y = 11$

17. $3x - 4y = 12$

18. $2x - 5y = 10$

19. $y + 1 = 0$

20. $y = 4x + 4$

21. $y = x + 2$

22. $x - 4 = -1$

23. $3y = 2x - 4$

24. $y = -3$

25. $4x = 3y + 8$

26. $3x + 5y = 6$

27. $2x + 7y = -4$

28. $2x + 3y = 1$

29. $x + 5 = 6$

30. $5x - 3y = -1$

31. $5x - 2y = 7$

32. $y - 3 = 1$

33. $3x + 4y = 7$

34. $\frac{2}{3}x - y = 4$

35. $x + \frac{3}{4}y = 6$

36. $2x + \frac{1}{2}y = 3$

37. $y + 2 = 3$

38. $\frac{2}{5}x - 3y = 5$

39. $5x = y + 2$

40. $4x = 3y - 5$

Graph each linear equation by locating the *x*-intercept and the *y*-intercept. See Examples 4 through 6.

41. $x + y = 6$

42. $x + y = 4$

43. $x - 2y = 8$

44. $x - 3y = 6$

45. $4x + y = 8$

46. $x + 3y = 9$

47. $x - 4y = -6$

48. $x - 6y = 3$

49. $y = 4x - 10$

50. $y = 2x - 9$

51. $3x - 2y = 6$

52. $5x + 2y = 10$

53. $2x + 3y = 12$

54. $3x + 7y = -21$

55. $3x - 7y = -21$

56. $3x + 2y = 15$

57. $5x + 3y = 7$

58. $2x + 3y = 5$

59. $y = \frac{1}{2}x - 4$

60. $y = -\frac{1}{3}x + 3$

61. $\frac{2}{3}x - 3y = 4$

62. $\frac{1}{2}x + 2y = 3$

63. $\frac{1}{2}x - \frac{3}{4}y = 6$

64. $\frac{2}{3}x + \frac{4}{3}y = 8$

Applications

Solve.

65. ***Chemistry:*** The amount of potassium in a clear bottle of a popular sports drink declines over time when exposed to the UV lights found in most grocery stores. The amount of potassium in a container of this sports drink is given by the equation $y = -30x + 360$, where y represents the mg of potassium remaining after x days on the shelf. Find both the x-intercept and y-intercept, and interpret the meaning of each in the context of this problem.

66. ***Education:*** Mr. Adler has found that the grade each student gets in his Introductory Algebra course directly correlates with the amount of time spent doing homework, and is represented by the equation $y = 7x + 30$, where y represents the numerical score the student receives on an exam (out of 100 points) after spending x hours per week doing homework. Find the y-intercept and interpret its meaning in this context.

Writing & Thinking

67. Explain, in your own words, why it is sufficient to find the x-intercept and y-intercept to graph a line (assuming that they are not the same point).

68. Explain, in your own words, how you can determine if an ordered pair is a solution to an equation.

Objectives

A. Interpret the slope of a line as a rate of change.

B. Find the slope of a line given two points on the line.

C. Find the slopes of horizontal and vertical lines.

D. Graph a linear equation by finding the slope and *y*-intercept.

E. Use slope-intercept form to write the equation of a line given its slope and *y*-intercept.

10.3 Slope-Intercept Form

A The Meaning of Slope

If you ride a bicycle up a mountain road, you certainly know when the **slope** (a measure of steepness called the **grade** for roads) increases because you have to pedal harder. The contractor who built the road was aware of the slope because trucks traveling the road must be able to control their downhill speed and be able to stop in a safe manner. A carpenter given a set of house plans calling for a roof with a **pitch** of 7 : 12 knows that for every 7 feet of rise (vertical distance) there are 12 feet of run (horizontal distance). That is, the ratio of rise to run is $\frac{rise}{run} = \frac{7}{12}$.

Figure 1

Note that this ratio can be in units other than feet, such as inches or meters. (See Figure 2.)

$$\frac{rise}{run} = \frac{7 \text{ inches}}{12 \text{ inches}} = \frac{3.5 \text{ feet}}{6 \text{ feet}} = \frac{14 \text{ feet}}{24 \text{ feet}}$$

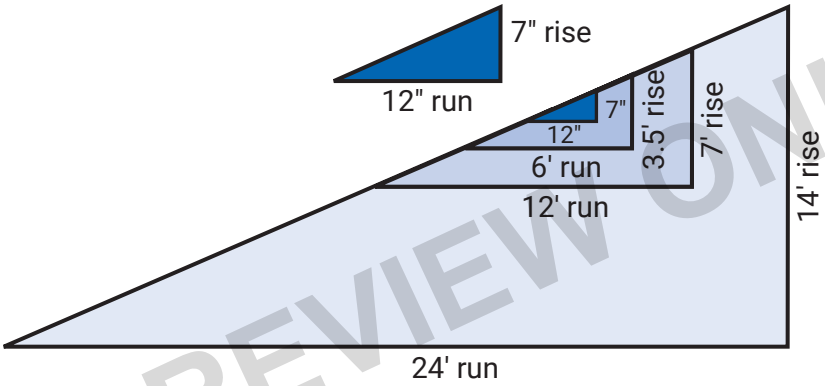
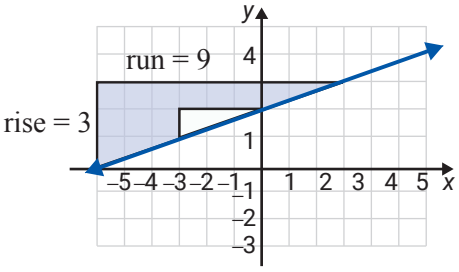


Figure 2

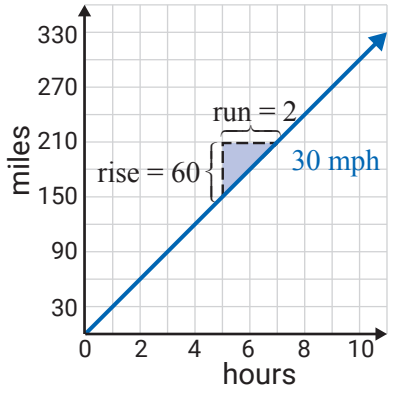
For a line, the **ratio of rise to run** is called the **slope of the line**. The graph of the linear equation $y = \frac{1}{3}x + 2$ is shown in Figure 3. What do you think is the slope of the line? Do you think that the slope is positive or negative? Do you think the slope might be $\frac{1}{3}$? $\frac{3}{1}$? 2?

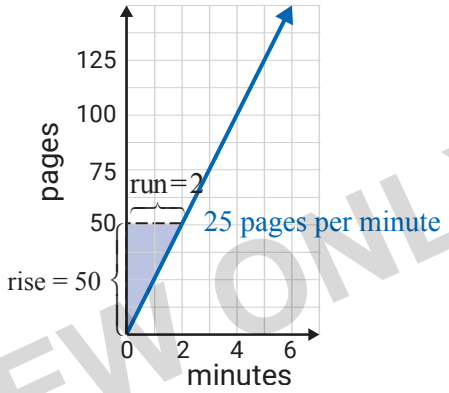


$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3}{9} = \frac{1}{3}$$

Figure 3

The concept of slope also relates to situations that involve **rate of change**. For example, the graphs in Figure 4 illustrate slope as miles per hour that a car travels and as pages per minute that a printer prints. (**Note:** Be sure to look at the scales on the axes when reading a graph.)





slope = rate of change

$$= \frac{60 \text{ miles}}{2 \text{ hours}}$$

$$= 30 \text{ mph}$$

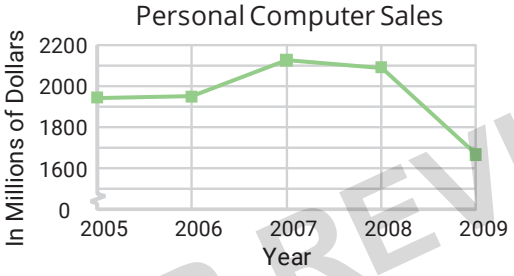
slope = rate of change

$$= \frac{50 \text{ pages}}{2 \text{ minutes}}$$

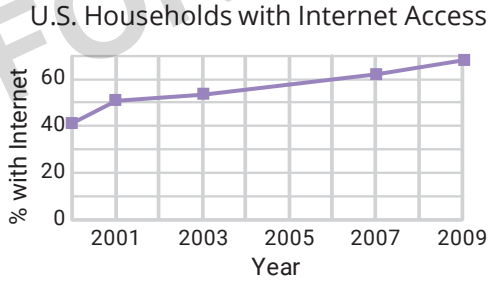
$$= 25 \text{ pages per minute}$$

Figure 4

In general, the ratio of a change in one variable (say y) to a change in another variable (say x) is called the **rate of change of y with respect to x**. Figure 5 shows how the rate of change (the slope) can change over periods of time.



Source: Consumer Electronics Association



Source: U.S. Dept. of Commerce

Figure 5

B Calculating the Slope

Note

In the notation P_1, 1 is called a **subscript** and P_1 is read "P sub 1." Similarly, P_2 is read "P sub 2." Subscripts are used in labeling and are not used in calculations.

Consider the line $y = 2x + 3$, and two points on the line $P_1(-2, -1)$ and $P_2(2, 7)$ as shown in Figure 6.

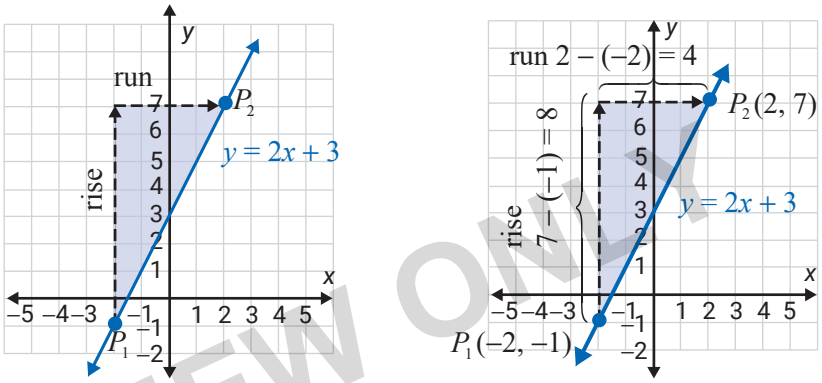


Figure 6

For the line $y = 2x + 3$ and using the points $(-2, -1)$ and $(2, 7)$ that are on the line,

$$\textbf{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}} = \frac{7-(-1)}{2-(-2)} = \frac{8}{4} = 2.$$

From similar illustrations and the use of subscript notation, we can develop the following formula for the slope of any line.

Slope

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on a line. The **slope** can be calculated as follows.

$$\textbf{slope} = m = \frac{\textit{rise}}{\textit{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note: The letter m is standard notation for representing the slope of a line.

FORMULA

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

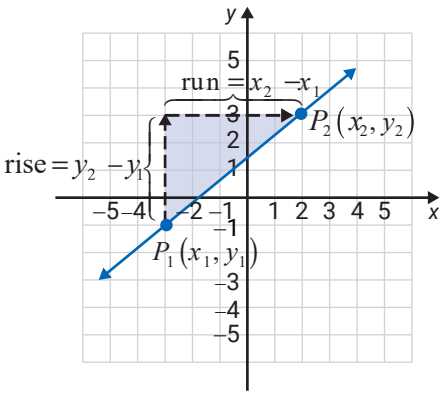


Figure 7

Example 1 Finding the Slope of a Line

Find the slope of the line that contains the points $(-1, 2)$ and $(3, 5)$, and then graph the line.

1. Find the slope of the line that contains the points $(3, -1)$ and $(4, 2)$, and then graph the line.

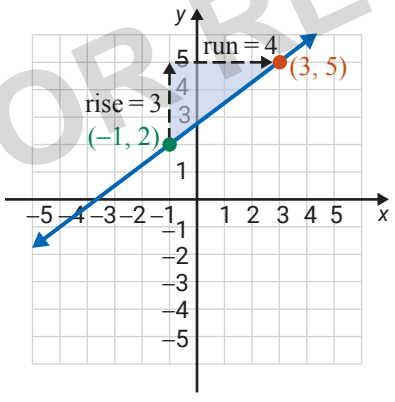
Solution

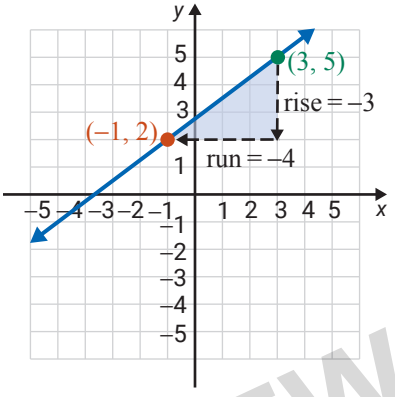
For (x_1, y_1), use $(-1, 2)$ and for (x_2, y_2), use $(3, 5)$.

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{3-(-1)} = \frac{3}{4}$$

Or, for (x_1, y_1) use $(3, 5)$ and for (x_2, y_2) use $(-1, 2)$.

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-5}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$$





Now work margin exercise 1.

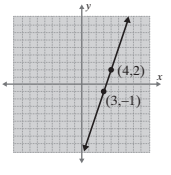
As we see in Example 1, **the slope is the same even if the order of the points is reversed**. The important part of the procedure is that **the coordinates must be subtracted in the same order in both the numerator and the denominator.**

In general,

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Answers

1. slope = 3



2. Find the slope of the line that contains the points $(0, 5)$ and $(4, 2)$, and then graph the line.

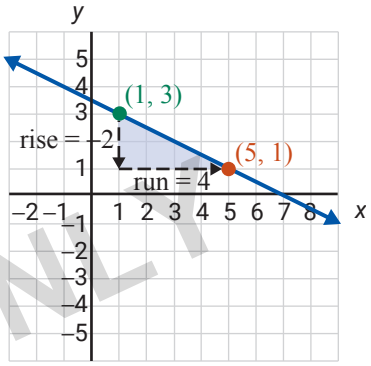
Example 2 Finding the Slope of a Line

Find the slope of the line that contains the points $(1, 3)$ and $(5, 1)$, and then graph the line.

Solution

For (x_1, y_1), use $(1, 3)$ and for (x_2, y_2), use $(5, 1)$.

$$\text{slope} = m = \frac{1-3}{5-1} = \frac{-2}{4} = -\frac{1}{2}$$



Now work margin exercise 2.

Positive and Negative Slope

Lines with **positive slope go up** (increase) as we move along the line from left to right.

Lines with **negative slope go down** (decrease) as we move along the line from left to right.

DEFINITION

C Slopes of Horizontal and Vertical Lines

In Section 10.1, we discussed the graphs of horizontal lines ($y = b$) and vertical lines ($x = a$), but the slopes of lines of these types were not discussed.

To find the slope of a horizontal line, such as $y = 3$, find two points on the line and substitute into the slope formula. Note that any two points on the line will have the same y-coordinate; namely, 3. Two such points are $(-2, 3)$ and $(5, 3)$. Using the two points in the formula for slope gives the following.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{5-(-2)} = \frac{0}{7} = 0$$

For any horizontal line, all of the y-values will be the same. Consequently, the formula for slope will always have 0 in the numerator. Therefore, **the slope of every horizontal line is 0**. (See Figure 8.)

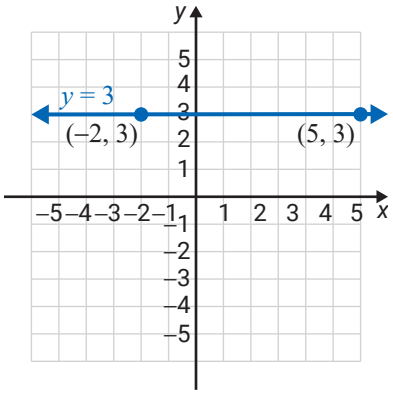
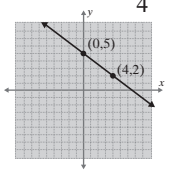


Figure 8

Answers

2. slope $= -\frac{3}{4}$



To find the slope of a vertical line, such as $x = 4$, find two points on the line and substitute into the slope formula. Now any two points on the line will have the same x-coordinate; namely, 4. Two such points are $(4, 1)$ and $(4, 6)$. The slope formula gives the following.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-1}{4-4} = \frac{5}{0}, \textbf{ which is undefined}$$

(Reminder, division by 0 is undefined, thus the slope is undefined.)

For any vertical line, all of the x-values will be the same. Consequently, the formula for slope will always have 0 in the denominator. Therefore, **the slope of every vertical line is undefined.** (See Figure 9.)

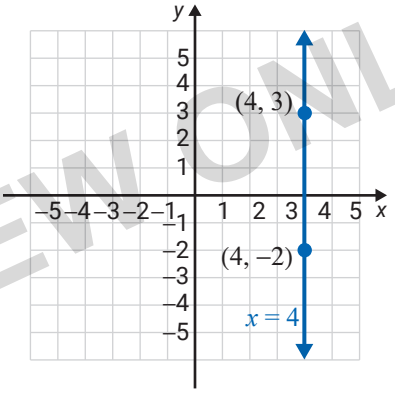


Figure 9

Horizontal and Vertical Lines

The following two general statements are true for horizontal and vertical lines.

1. For **horizontal lines** (of the form $y = b$), the **slope is 0**.
2. For **vertical lines** (of the form $x = a$), the **slope is undefined**.

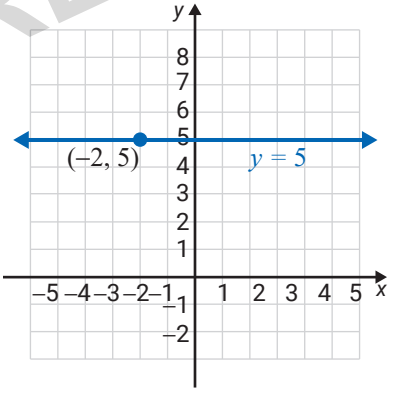
DEFINITION

Example 3 Finding the Slope of a Horizontal Line

Find the equation and slope of the horizontal line through the point $(-2, 5)$.

Solution

The equation is $y = 5$ and the slope is 0.



Now work margin exercise 3.

3. Find the equation and slope of the horizonal line through the point $(3, -2)$.

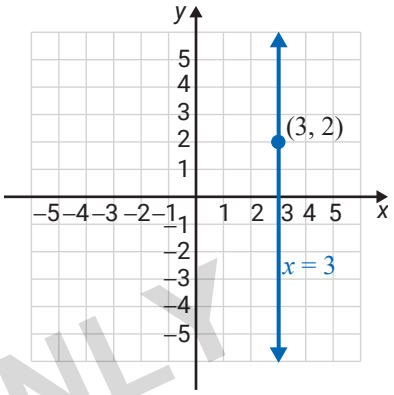
Answers

3. $y = -2$; slope is 0

4. Find the equation and slope of the vertical line through the point $(2,4)$.

Example 4 Finding the Slope of a Vertical Line

Find the equation and slope of the vertical line through the point $(3, 2)$.



Solution

The equation is $x = 3$ and the slope is undefined.

Now work margin exercise 4.

D Slope-Intercept Form

There are certain relationships between the coefficients in the equation of a line and the graph of that line. For example, consider the equation

$$y = 5x - 7.$$

First, find two points on the line and calculate the slope. The points $(0, -7)$ and $(2, 3)$ both satisfy the equation.

$$\text{slope} = m = \frac{3-(-7)}{2-0} = \frac{10}{2} = 5$$

Observe that the slope, $m = 5$, is the same as the coefficient of x in the equation $y = 5x - 7$. This is not just a coincidence. In fact, if a linear equation is solved for y, then the coefficient of x will always be the slope of the line.

The Slope *m*

For an equation in the form $\boldsymbol{y = mx + b}$, the slope of the line is m.

DEFINITION

For the line $y = mx + b$, the point where $x = 0$ is the point where the line crosses the y-axis. Recall that this point is called the ***y*-intercept**. By letting $x = 0$, we get

$$y = mx + b$$
$$y = m \cdot 0 + b$$
$$y = b.$$

Thus, the point $(0, b)$ is the y-intercept. The concepts of slope and y-intercept lead to the following definition.

Answers

4. $x = 2$; slope is undefined

Slope-Intercept Form

$y = mx + b$ is called the **slope-intercept** form for the equation of a line, where m is the **slope** and $(0, b)$ is the ***y*-intercept**.

DEFINITION

As illustrated in Example 4, an equation in **standard form**

$$Ax + By = C \text{ with } B \neq 0$$

can be written in slope-intercept form by solving for y.

Example 5 Using Slope and the *y*-Intercept to Graph a Line

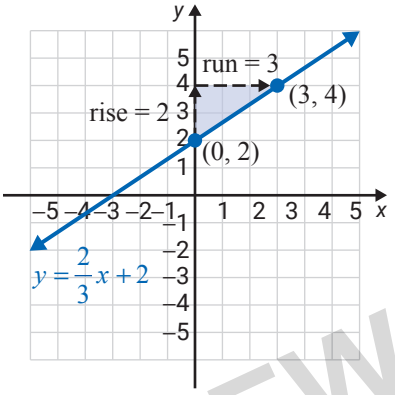
Find the slope and y-intercept of $-2x + 3y = 6$, and graph the line.

5. Find the slope and y-intercept of $-4x + 2y = 12$, and graph the line.

Solution

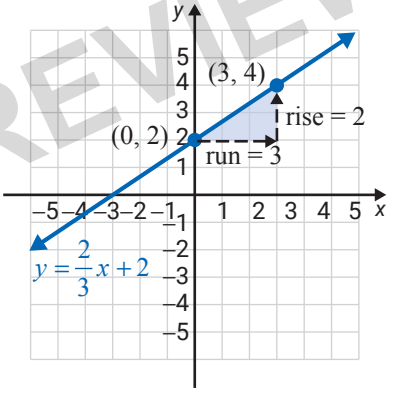
Solve for y.

$$\begin{aligned} -2x + 3y &= 6 \\ 3y &= 2x + 6 \\ \frac{3y}{3} &= \frac{2x}{3} + \frac{6}{3} \\ y &= \frac{2}{3}x + 2 \end{aligned}$$



Thus $m = \frac{2}{3}$, which is the slope, and b is 2, making the y-intercept equal to $(0, 2)$.

As shown in the graph, if we "rise" 2 units up and "run" 3 units to the right **from the *y*-intercept** $(0, 2)$, we locate another point $(3, 4)$. The line can be drawn through these two points.

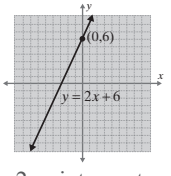


Note: As shown in the second graph, we could also first "run" 3 units right and "rise" 2 units up from the y-intercept to locate the point $(3, 4)$ on the graph.

Now work margin exercise 5.

Answers

5.



$m = 2$; y-intercept $= (0, 6)$

6. Find the slope and y-intercept of $3x+2y=-10$, and graph the line.

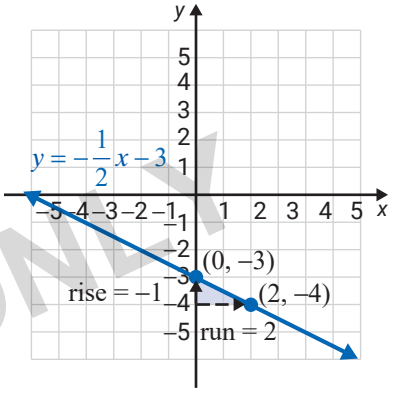
Example 6 Using Slope and the y-Intercept to Graph a Line

Find the slope and y-intercept of $x+2y=-6$, and graph the line.

Solution

Solve for y.

$$x+2y=-6$$
$$2y=-x-6$$
$$\frac{2y}{2}=\frac{-x}{2}-\frac{6}{2}$$
$$y=-\frac{1}{2}x-3$$



Thus $m=-\frac{1}{2}$, which is the slope, and b is -3, making the y-intercept equal to $(0,-3)$.

We can treat $m=-\frac{1}{2}$ as $m=\frac{-1}{2}$ and the "rise" as -1 and the "run" as 2. Moving from $(0,-3)$ as shown in the graph, we locate another point $(2,-4)$ on the graph and draw the line.

Now work margin exercise 6.

E Finding Equations of Lines Given the Slope and the y-Intercept

7. Find the equation of the line through the point $(0,-3)$ with a slope of $\frac{2}{3}$.

Example 7 Finding Equations Given the Slope and the y-Intercept

Find the equation of the line through the point $(0,-2)$ with slope $\frac{1}{2}$.

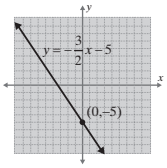
Solution

Because the x-coordinate is 0, we know that the point $(0,-2)$ is the y-intercept. So $b=-2$. The slope is $\frac{1}{2}$. So $m=\frac{1}{2}$. Substituting in slope-intercept form $y=mx+b$ gives the result $y=\frac{1}{2}x-2$.

Now work margin exercise 7.

Answers

6.



$m=-\frac{3}{2}$;
y-intercept $=(0,-5)$

7. $y=\frac{2}{3}x-3$

10.3 Exercises

Concept Check

Fill-in-the-Blank. Complete the sentences using information found in this chapter.

1. The slope of a line is the ratio of rise to _____.
2. Another name for slope is the rate of _____.
3. A line that rises (increases) from left to right has a/an _____ slope.
4. The slope of every vertical line is _____.
5. The slope of every horizontal line is _____.
6. In the equation $y = mx + b$, m represents the _____ and $(0, b)$ represents the _____.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so the statement will be true. (Note: There may be more than one acceptable change.)

7. If the y-intercept and the slope of a line are given, there is enough information to write the equation of the line.
8. When using the slope formula, the slope of a line changes if the order of the points is reversed.
9. A line that falls (decreases) from left to right has a negative slope.
10. The line that represents the equation $y = 2x + 4$ has a y-intercept of $(0, 4)$.

Practice

Find the slope of the line determined by each pair of points. See Examples 1 and 2.

1. $(2, 4); (1, -1)$
2. $(1, -2); (1, 4)$
3. $(-6, 3); (1, 2)$
4. $(-3, 7); (4, -1)$
5. $(-5, 8); (3, 8)$
6. $(-2, 3); (-2, -1)$
7. $(5, 1); (3, 0)$
8. $(0, 0); (-2, -3)$
9. $\left(\frac{3}{4}, \frac{3}{2}\right); (1, 2)$
10. $\left(4, \frac{1}{2}\right); (-1, 2)$
11. $\left(\frac{3}{2}, \frac{4}{5}\right); \left(-2, \frac{1}{10}\right)$
12. $\left(\frac{7}{2}, \frac{3}{4}\right); \left(\frac{1}{2}, -3\right)$

Determine whether each equation represents a horizontal line or vertical line and give its slope. Graph the line. See Examples 3 and 4.

13. $y = 5$

14. $y = -2$

15. $x = -3$

16. $x = 1.7$

17. $3y = -18$

18. $4x = 2.4$

19. $-3x + 21 = 0$

20. $2y + 5 = 0$

Write each equation in slope-intercept form. Find the slope and *y*-intercept, and then use them to draw the graph. See Examples 5 and 6.

21. $y = 2x - 1$

22. $y = 3x - 4$

23. $y = 5 - 4x$

24. $y = 4 - x$

25. $y = \frac{2}{3}x - 3$

26. $y = \frac{2}{5}x + 2$

27. $x + y = 5$

28. $x - 2y = 6$

29. $x + 5y = 10$

30. $4x + y = 0$

31. $4x + y + 3 = 0$

32. $2x + 7y + 7 = 0$

33. $2y - 8 = 0$

34. $3y - 9 = 0$

35. $2x = 3y$

36. $4x = y$

37. $3x + 9 = 0$

38. $4x + 7 = 0$

39. $5x - 6y = 18$

40. $3x + 6 = 6y$

41. $5 - 3x = 4y$

42. $5x = 11 - 2y$

43. $6x + 4y = -8$

44. $7x + 2y = 4$

45. $6y = -6 + 3x$

46. $4x = 3y - 7$

47. $5x - 2y + 5 = 0$

48. $6x + 5y = -15$

In reference to the equation $y = mx + b$, sketch the graphs of three lines for each of the two characteristics listed below.

49. $m > 0$ and $b > 0$

50. $m < 0$ and $b > 0$

51. $m > 0$ and $b < 0$

52. $m < 0$ and $b < 0$

Find an equation in slope-intercept form for the line passing through the given point with the given slope. See Example 7.

53. $(0, 3); m = -\frac{1}{2}$

54. $(0, 2); m = \frac{1}{3}$

55. $(0, -3); m = \frac{2}{5}$

56. $(0, -6); m = \frac{4}{3}$

57. $(0, -5)$; $m = 4$

58. $(0, 9)$; $m = -1$

59. $(0, -4)$; $m = 1$

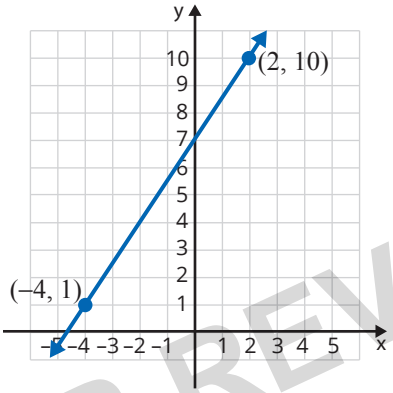
60. $(0, 6)$; $m = -5$

61. $(0, -3)$; $m = -\frac{5}{6}$

62. $(0, -1)$; $m = -\frac{3}{2}$

The graph of a line is shown with two points labeled. Find **a.** the slope, **b.** the *y*-intercept (if there is one), and **c.** the equation of the line in slope-intercept form.

63.



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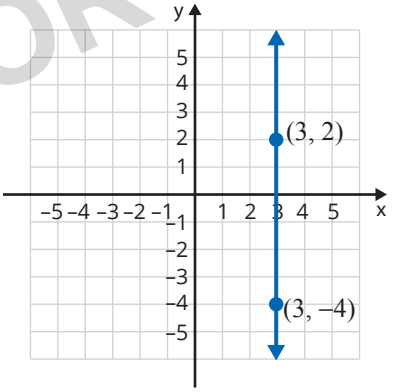

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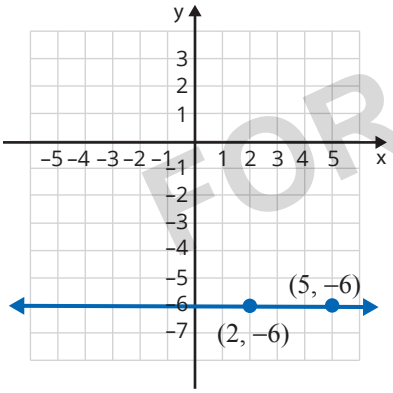

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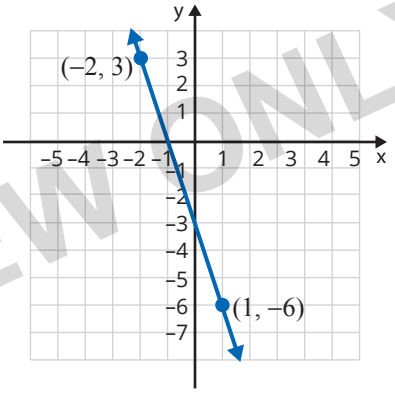

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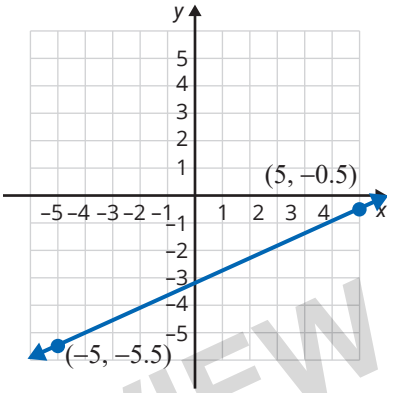

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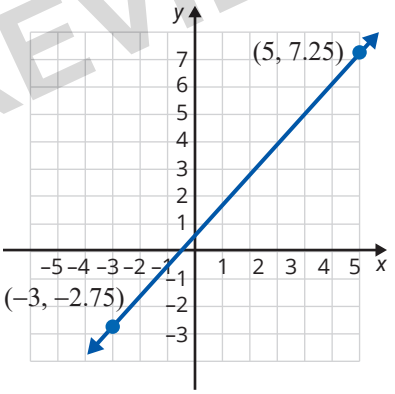

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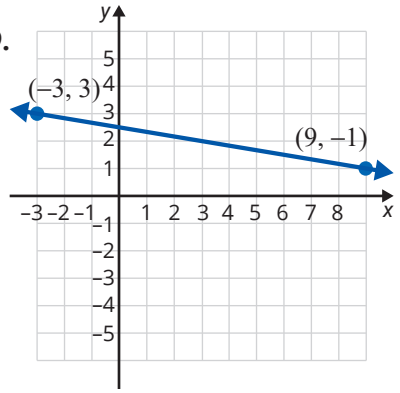

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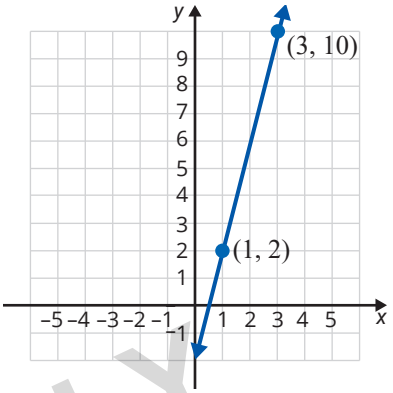


Points are said to be **collinear** if they lie on a straight line. If points are collinear, then the slope of the line through any two of them must be the same (because the line is the same line). Use this idea to determine whether or not the three points in each of the sets are collinear.

71. $\{(-1,3),(0,1),(5,-9)\}$

72. $\{(-2,-4),(0,2),(3,11)\}$

73. $\{(-2,0),(0,30),(1.5,5.25)\}$

74. $\{(-1,-7),(1,1),(2.5,7)\}$

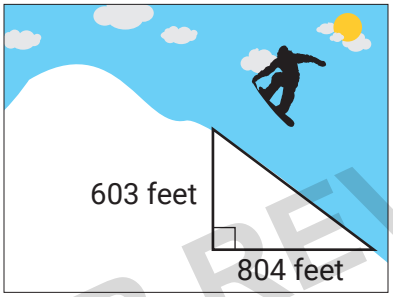
75. $\left\{\left(\frac{2}{3},\frac{1}{2}\right),\left(0,\frac{5}{6}\right),\left(-\frac{3}{4},\frac{29}{24}\right)\right\}$

76. $\left\{\left(\frac{3}{2},-\frac{1}{3}\right),\left(0,\frac{1}{6}\right),\left(-\frac{1}{2},\frac{3}{4}\right)\right\}$

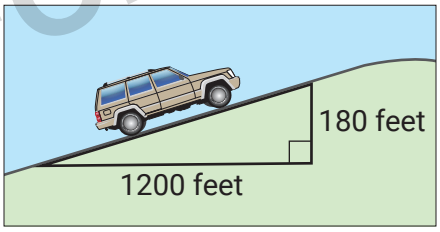
Applications

Solve.

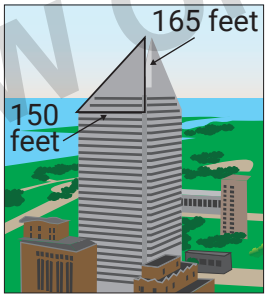
77. Find the slope of the ski slope.



78. Find the slope of the road.

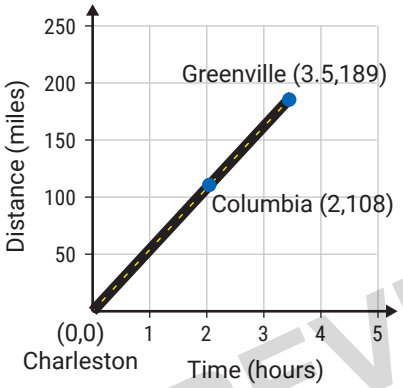


79. Find the slope of the roof of the skyscraper.

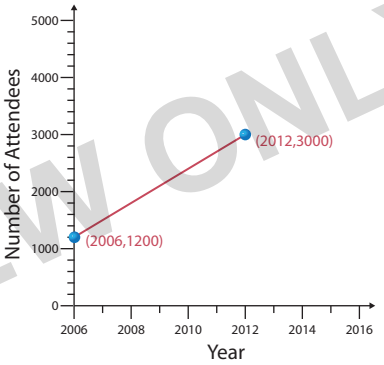


80. Find the slope of the larger sail on the sailboat.

81. ***Travel:*** A car travels from Charleston to Greenville. Its distance related to time traveled is given on the following graph. Find the average speed of the car in miles per hour from Columbia to Greenville.



82. ***Festivals:*** The attendance at Smithville's Spring Festival has been increasing steadily as shown in the following graph. Find the average increase in attendees per year. How many people do you predict will attend the festival in 2016?



83. ***Purchases:*** John bought his new car for $35,000 in the year 2014. He knows that the value of his car has depreciated linearly. If the value of the car in 2017 was $23,000, what was the annual rate of depreciation of his car? Show this information on a graph. (When graphing, use years as the x-coordinates and the corresponding values of the car as the y-coordinates.)

84. ***Cell Phones:*** The number of people in the United States with mobile cellular phones was about 198 million in 2011 and about 232 million in 2016. If the growth in the usage of mobile cellular phones was linear, what was the approximate rate of growth per year from 2011 to 2016. Show this information on a graph. (When graphing, use years as the x-coordinates and the corresponding numbers of users as the y-coordinates.)

Source: https://www.statista.com/statistics/231612/number-of-cell-phone-users-usa/

85. ***Internet:*** The given table shows the estimated number of internet users from 2010 to 2014. The number of users for each year is shown in millions.

Year	Internet users (in millions)
2010	222
2011	218
2012	250
2013	267
2014	279

Source: https://www.statista.com/statistics/276445/number-of-internet-users-in-the-united-states/

a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

86. ***Population:*** The following table shows the urban growth from 1850 to 2000 in New York, NY.

Year	Population
1850	515,547
1900	3,437,202
1950	7,891,957
2000	8,008,278

Source: U.S. Census Bureau

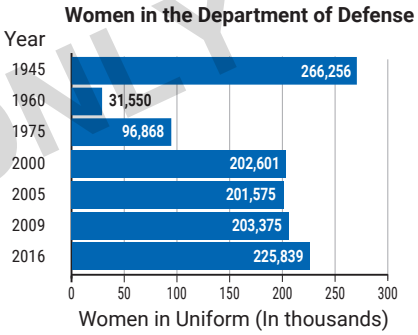
a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

87. ***Military:*** The following graph shows the number of female active duty military personnel over a span from 1945 to 2016. The number of women listed includes both officers and enlisted personnel from the Army, the Navy, the Marine Corps, and the Air Force.



Source: U.S. Dept. of Defense

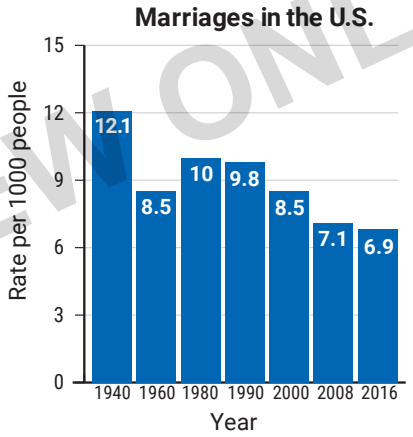
a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

88. ***Marriage:*** The following graph shows the rates of marriage per 1000 people in the U.S. over a span from 1940 to 2016.



Source: U.S. National Center for Health Statistics

a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

Collaborative Learning

89. The class should be divided into teams of 2 or 3 students. Each team will need access to a digital camera, a printer, and a ruler.

a. Take pictures of 8 things with a defined slope. (**Suggestions:** A roof, a stair railing, a beach umbrella, a crooked tree, etc. Be creative!)

b. Print each picture.

c. Use a ruler to draw a coordinate system on top of each picture. You will probably want to use increments of in. or cm, depending on the size of your picture.

d. Identify the line in each picture whose slope you are calculating and then use the coordinate systems you created to identify the coordinates of two points on each line.

e. Use the points you just found to calculate the slope of the line in each picture.

f. Share your findings with the class.

Writing & Thinking

90. a. Explain in your own words why the slope of a horizontal line must be 0.

b. Explain in your own words why the slope of a vertical line must be undefined.

91. a. Describe the graph of the line $y = 0$.

b. Describe the graph of the line $x = 0$.

92. In the formula $y = mx + b$, explain the meaning of m and the meaning of b.

93. The slope of a road is called a **grade**. A steep grade is cause for truck drivers to have slow speed limits in mountains. What do you think that a "grade of 12%" means? Draw a picture of a right triangle that would indicate a grade of 12%.

Objectives

A. Know the characteristics of an exponential function.

B. Recognize exponential growth functions.

C. Recognize exponential decay functions.

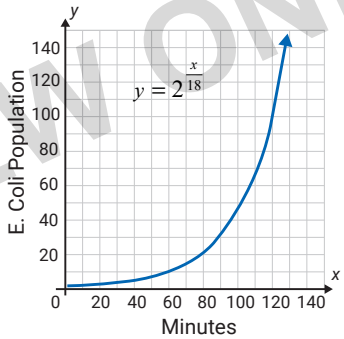
D. Use exponential growth and decay functions to solve applications.

E. Use exponential functions and the number *e* to solve compound interest applications.

17.3 Exponential Functions

A Introduction to Exponential Functions

You may have read that the population of the world is growing exponentially or studied the exponential growth of bacteria in a biology class. Radioactive materials decay exponentially and never actually disappear. The graph in Figure 1 illustrates that exponential growth has a relatively slow beginning and then builds at an exceedingly rapid rate. This can be extremely important to a doctor trying to curb the growth of "bad" bacteria in a patient.



The population of certain strains of E. coli doubles every 18 minutes under optimal conditions.

Figure 1

Quadratic functions have a variable base and a constant exponent, as in $f(x) = x^2$. However, in **exponential functions**, the base is constant and the variable is in the exponent, as in $f(x) = 2^x$. As we will see, these two types of functions have major differences in their characteristics. Exponential functions are defined as follows.

Notes

The two conditions $b > 0$ and $b \neq 1$ in the definition are important. We must have $b > 0$ so that b^x is defined for all real x. For example, we do not consider $y = (-2)^x$ to be an exponential function because $(-2)^{\frac{1}{2}}$ is not a real number. Also, $b \neq 1$. Because the function $y = 1^x = 1$ for all real x, this function is not considered to be an exponential function.

Exponential Functions

An **exponential function** is a function of the form

$$f(x) = b^x,$$

where $b > 0$, $b \neq 1$, and x is any real number.

DEFINITION

Examples of exponential functions are

$$f(x) = 2^x, \quad f(x) = 3^x, \quad \text{and} \quad y = \left(\frac{1}{3}\right)^x.$$

B Exponential Growth

The following table of values and the graphs of the corresponding points give a very good idea of what the graph of the **exponential growth** function $y = 2^x$ looks like (see Figure 2a.). Because we know that 2^x is defined for all real exponents, points such as $(\sqrt{2}, 2^{\sqrt{2}})$, $(\pi, 2^{\pi})$, and $(\sqrt{5}, 2^{\sqrt{5}})$ are on the graph. The graph for $f(x) = 2^x$ is a smooth curve, as shown in Figure 2b.

x	$y = 2^x$
3	$2^3 = 8$
2	$2^2 = 4$
1	$2^1 = 2$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2} \approx 1.4142$
0	$2^0 = 1$
$-\frac{1}{2}$	$2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.7071$
-1	$2^{-1} = \frac{1}{2}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-4	$2^{(-4)} = \frac{1}{2^4} = \frac{1}{16}$

Domain $= (-\infty, \infty)$ Range $= (0, \infty)$

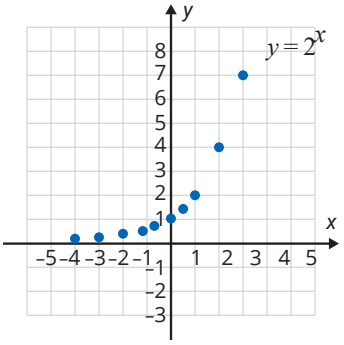


Figure 2a

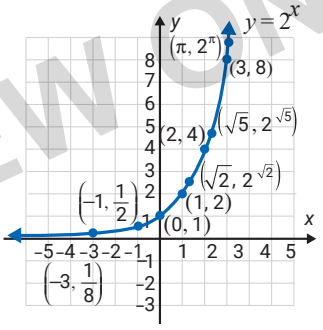


Figure 2b

Figure 3 shows a table of values and the graph of the function $y = 3^x$. Note that the graphs of $y = 2^x$ and $y = 3^x$ are quite similar, but that the graph of $y = 3^x$ rises faster. That is, the **exponential growth is faster if the base is larger**.

x	$y = 3^x$
2	$3^2 = 9$
1	$3^1 = 3$
0	$3^0 = 1$
-1	$3^{-1} = \frac{1}{3} \approx 0.3333$
-3	$3^{-3} = \frac{1}{27} \approx 0.0370$

Domain $= (-\infty, \infty)$ Range $= (0, \infty)$

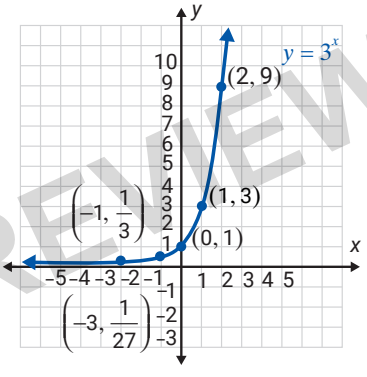


Figure 3

Notice that in both graphs the curves tend to get **very close** to the line $y = 0$ (the x-axis) without ever touching the x-axis. When this happens, the line is called an **asymptote**. If the line is horizontal, as in the cases of exponential growth and (as we will see) exponential decay, the line is called a **horizontal asymptote**. We say that the curve (or function) approaches the line **asymptotically**. In mathematics, this phenomenon happens frequently.

C Exponential Decay

Now consider the **exponential decay** function $f(x)=\left(\frac{1}{2}\right)^x$. The table and the graph of the corresponding points shown in Figure 4 indicate the nature of the graph of this function.

x	$y=\left(\frac{1}{2}\right)^x=2^{-x}$
-3	$2^{-(-3)}=2^3=8$
-2	$2^{-(-2)}=2^2=4$
-1	$2^{-(-1)}=2^1=2$
$-\frac{1}{2}$	$2^{-\left(-\frac{1}{2}\right)}=2^{\frac{1}{2}}=\sqrt{2}\approx 1.4142$
0	$2^{-0}=2^0=1$
$\frac{1}{2}$	$2^{-\frac{1}{2}}=\frac{1}{\sqrt{2}}\approx 0.7071$
1	$2^{-1}=\frac{1}{2}$
2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$
3	$2^{-3}=\frac{1}{2^3}=\frac{1}{8}$

Domain $=(-\infty,\infty)$ Range $=(0,\infty)$

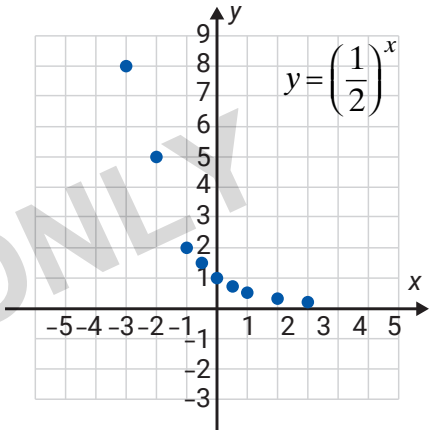


Figure 4

Notes

Because $\frac{1}{2}=2^{-1}$, $\frac{1}{3}=3^{-1}$, $\frac{1}{4}=4^{-1}$, and so on, for fractions between 0 and 1, we can write an exponential function with a fractional base between 0 and 1 (these are exponential decay functions) in the form of an exponential function with a base greater than 1 and a negative exponent. Thus, we write

$$y=\left(\frac{1}{2}\right)^x=\left(2^{-1}\right)^x=2^{-x}$$

and

$$y=\left(\frac{1}{3}\right)^x=\left(3^{-1}\right)^x=3^{-x}.$$

Figures 5a and 5b show the complete graphs of the two exponential decay functions

$$y=\left(\frac{1}{2}\right)^x=2^{-x} \text{ and } y=\left(\frac{1}{3}\right)^x=3^{-x}.$$

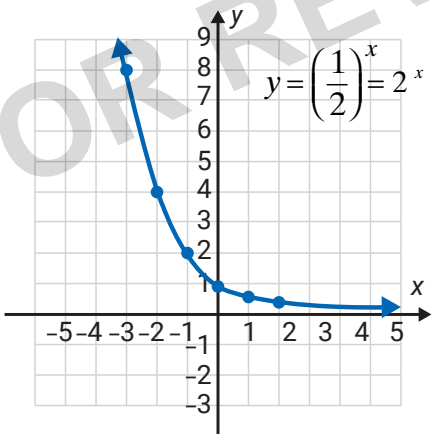


Figure 5a

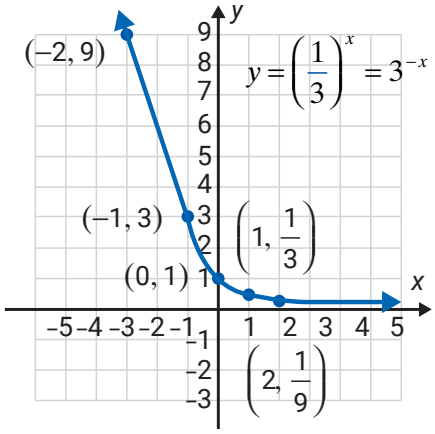


Figure 5b

Note again, in Figures 5a and 5b, that the line $y=0$ (the x-axis) is a **horizontal asymptote** for both curves. For exponential growth the curves approach the

asymptote as x moves further in the negative direction, as in Figures 2b and 3. For exponential decay, the graph approaches the asymptote as x moves further in the positive direction as in Figures 5a and b.

The following general concepts are helpful in understanding the graphs and the nature of exponential functions, both exponential growth and exponential decay.

General Concepts of Exponential Functions

For $b > 1$:

1. $b^x > 0$
2. b^x increases to the right and is called an **exponential growth function**.
3. $b^0 = 1$, so $(0,1)$ is the y-intercept.
4. b^x approaches the x-axis for negative values of x. (The x-axis is a horizontal asymptote. **See Figure 3**.)

For $0 < b < 1$:

1. $b^x > 0$
2. b^x decreases to the right and is called an **exponential decay function**.
3. $b^0 = 1$, so $(0,1)$ is the y-intercept.
4. b^x approaches the x-axis for positive values of x. (The x-axis is a horizontal asymptote. **See Figure 4**.)

As with all functions, exponential functions can be multiplied by constants, shifted horizontally, and shifted vertically. (**Note:** We did this with parabolas in Chapter 16 and will be more detailed on this topic in Chapter 18.) Thus,

$$y = a \cdot b^{r \cdot x}, \quad y = b^{x-h}, \quad \text{and} \quad y = b^x + k,$$

and various combinations of these expressions are all exponential functions.

D Applications of Exponential Functions

Exponential functions are related to many practical applications, among which are bacterial growth, radioactive decay, compound interest, and light absorption. For example, a bacteria culture kept at a certain temperature may grow according to the exponential function

$y = y_0 \cdot 2^{0.5t}$, where t = time in hours and
y_0 = amount of bacteria present when $t = 0$.
(y_0 is called the **initial value** of y.)

1. Using the same function and initial value given in Example 1, how many bacteria will be present at the end of 36 hours?
$t = 36$ hours and $y_0 = 10{,}000$

Example 1 Application: Calculating Bacterial Growth

A scientist has 10,000 bacteria present when $t = 0$. She knows the bacteria grow according to the function $y = y_0 \cdot 2^{0.5t}$, where t is measured in hours. How many bacteria will be present at the end of one day?

Solution

Substitute $t = 24$ hours and $y_0 = 10{,}000$ into the function.

$$\begin{aligned} y &= 10{,}000 \cdot 2^{0.5(24)} = 10{,}000 \cdot 2^{12} \\ &= 10{,}000(4096) \\ &= 40{,}960{,}000 \\ &= 4.096 \times 10^7 \end{aligned}$$

At the end of one day, there will be 40,960,000 bacteria.

CALCULATORS

Calculating Bacterial Growth

You could also use your calculator to calculate the result by entering the numbers as shown in the following display. Press ENTER to get the result.

```
10000*2^(0.5*24)
          40960000
```

Now work margin exercise 1.

2. Determine the exponential growth function that fits the following information: $y_0 = 7000$ bacteria with 112,000 bacteria present after 4 days.

Example 2 Application: Calculating Bacterial Growth

Use the formula for exponential growth, $y = y_0 b^t$, to determine the exponential function that fits the following information: $y_0 = 5000$ bacteria with 135,000 bacteria present after 3 days.

Solution

Use $y = y_0 b^t$ where t is measured in days. Substitute 135,000 for y, 3 for t, and 5000 for y_0, then solve for b.

$$\begin{aligned} 135{,}000 &= 5000b^3 \\ 27 &= b^3 \\ \sqrt[3]{(27)} &= \sqrt[3]{(b^3)} \\ 3 &= b \end{aligned}$$

The function is $y = 5000 \cdot 3^t$.

Now work margin exercise 2.

E Compound Interest and the Number *e*

The topic of compound interest (interest paid on interest) leads to a particularly interesting (and useful) exponential function. The formula $A = P(1+r)^t$ can be used for finding the value (amount) accumulated when a principal P is invested and interest is compounded once a year. If compounding is performed more than once a year, we use the following formula to find A.

Answers

1. $y = 2{,}621{,}440{,}000$ or 2.62144×10^9
2. $y = 7000 \cdot 2^t$

Compound Interest

Compound interest on a principal P invested at an annual interest rate r (in decimal form) for t years that is compounded n times per year can be calculated using the following formula where A is the amount accumulated.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

FORMULA

Example 3 Application: Calculating Compound Interest

If P dollars are invested at a rate of interest r (in decimal form) compounded annually (once a year, $n = 1$) for t years, the formula for the amount A becomes $A = P\left(1+\frac{r}{1}\right)^{1\cdot t} = P(1+r)^t$. Find the value of \$1000 invested at $r = 6\% = 0.06$ for 3 years.

3. Find the value of \$2000 invested at 5% for 2 years if the interest is compounded annually.

Solution

We have $P = 1000$, $r = 0.06$, and $t = 3$.

$$\begin{aligned} A &= 1000(1+0.06)^3 \\ &= 1000(1.06)^3 \\ &= 1000(1.191016) \approx 1191.02 \end{aligned}$$

The account will have \$1191.02 invested in it after 3 years.

Now work margin exercise 3.

CALCULATORS

Caclulating Compund Interest

To use your calculator to evaluate $(1.06)^3$, enter 1.06, press the ^ key, enter 3, then press the ENTER key.

Example 4 Application: Calculating Compound Interest

What will be the value of a principal investment of \$1000 invested at 6% for 3 years if interest is compounded monthly (12 times per year)?

4. Find the value of \$2000 invested at $r = 5\%$ for 2 years if the interest is compounded monthly.

Solution

Use the formula for compound interest.

We have $P = 1000$, $r = 0.06$, $n = 12$, and $t = 3$.

$$\begin{aligned} A &= 1000\left(1+\frac{0.06}{12}\right)^{12(3)} \\ &= 1000(1+0.005)^{36} \\ &= 1000(1.005)^{36} \\ &= 1000(1.196680524...) \quad \text{Using a calculator} \\ &\approx 1196.68 \end{aligned}$$

The value of the account after 3 years will be \$1196.68.

Now work margin exercise 4.

Answers

3. $A = \$2205$

4. $A = \$2209.88$

5. Find the value of \$2000 invested at $r = 5\%$ for 2 years if the interest is compounded daily.

Example 5 Application: Calculating Compound Interest

Find the value of A if \$1000 is invested at 6% for 3 years and interest is compounded daily (365 times per year).

Solution

Use the formula for compound interest

We have $P = 1000$, $r = 0.06$, $n = 365$, and $t = 3$.

$$\begin{aligned} A &= 1000\left(1+\frac{0.06}{365}\right)^{365(3)} \\ &= 1000(1.000164384...)^{1095} \\ &= 1000(1.197199652...) \quad \text{Using a calculator} \\ &\approx 1197.20 \end{aligned}$$

After three years, there will be \$1196.20 in the account.

Now work margin exercise 5.

Examples 3, 4, and 5 illustrate the effects of compounding interest more frequently over 3 years. The formula gives the following results.

$A = \$1191.02$ for $n = 1$ (once a year)

$A = \$1196.68$ for $n = 12$ (quarterly)

$A = \$1197.20$ for $n = 365$ (daily)

These numbers might not seem very dramatic, only a difference of \$6.18 for 3 years; but, if you use your calculator, in 20 years you will see a difference of \$112.65 for a \$1000 investment. An investment of \$10,000 for 20 years at 9% will show a difference of \$4438.94. The results show that more frequent compounding will result in higher income.

If interest is **compounded continuously** (which is even faster than every second), then the irrational number e $(e \approx 2.718)$ can be shown to be the base of the corresponding exponential function for calculating interest. Table 1 shows how the expression $\left(1+\frac{1}{n}\right)^n$ changes as n takes on larger and larger values. The number e is the **limit** (or **limiting value**) of the expression "as n approaches infinity" $(n \to \infty)$ and we write $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. Study the following table to help understand the ideas.

Answers
5. $A = \$2210.33$

Values of the expression $\left(1+\frac{1}{n}\right)^n$ as n approaches infinity ($n \to \infty$)

n	$\left(1+\frac{1}{n}\right)$	$\left(1+\frac{1}{n}\right)^n$
1	$\left(1+\frac{1}{1}\right)=2$	$(2)^1=2$
2	$\left(1+\frac{1}{2}\right)=1.5$	$(1.5)^2=2.25$
5	$\left(1+\frac{1}{5}\right)=1.2$	$(1.2)^5=2.48832$
10	$\left(1+\frac{1}{10}\right)=1.1$	$(1.1)^{10}\approx 2.59374246$
100	$\left(1+\frac{1}{100}\right)=1.01$	$(1.01)^{100}\approx 2.704813829$
1000	$\left(1+\frac{1}{1000}\right)=1.001$	$(1.001)^{1000}\approx 2.716923932$
10,000	$\left(1+\frac{1}{10,000}\right)=1.0001$	$(1.0001)^{10,000}\approx 2.718145927$
100,000	$\left(1+\frac{1}{100,000}\right)=1.00001$	$(1.00001)^{100,000}\approx 2.718268237$
$\downarrow$		$\downarrow$
∞		$e=2.718281828459\ldots$

Table 1

This gives us the following definition of e. (Be aware that these are very sophisticated mathematical concepts and it will take some careful reading to understand how to arrive at e and the formula $A = Pe^{rt}$.)

The Number e

The number e is defined to be

$$\boldsymbol{e = 2.718281828459\ldots}$$

Note: The number e is on the TI-84 Plus calculator above the divide key.

DEFINITION

As we know, the formula for compound interest is

$$A = P\left(1+\frac{r}{n}\right)^{nt}.$$

This formula for compound interest takes a different form when interest is **compounded continuously**. The new form involves the number e and is stated here.

Continuously Compounded Interest

Continuously compounded interest on a principal P invested at an annual interest rate r for t years can be calculated using the following formula where A is the amount accumulated.

$$A = Pe^{rt}$$

FORMULA

As illustrated in Example 6, a calculator is needed to use the formula for continuously compounded interest.

6. Find the value of \$1500 invested at 5.5% for 4 years if interest is compounded continuously.

Example 6 Using a Graphing Calculator to Calculate Continuously Compounded Interest

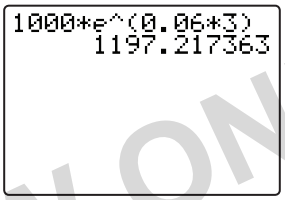
Find the value of \$1000 invested at 6% for 3 years if interest is compounded continuously. (In this case, $P = \$1000$, $r = 6\% = 0.06$, and $t = 3$.)

Solution

Press [2nd] and [LN] and e^(will appear on the display.

To find the value of $A = Pe^{rt} = 1000e^{0.06 \cdot 3}$ enter the numbers as shown and press [ENTER] to get the result.

The entire exponent must be in parentheses.



Thus, the value of \$1000 compounded continuously at 6% for 3 years will be \$1197.22. (Note that from Example 4 there is only a 54 cent gain in A when \$1000 is compounded continuously instead of monthly at 6% for 3 years.)

Now work margin exercise 6.

Answers
6. \$1869.12

17.3 Exercises

Concept Check

Fill-in-the-Blank. Complete each sentence using information found in this section.

1. In an exponential function, the base is a/an _____ and the exponent is a/an _____.
2. Exponential growth is faster if the base is _____.
3. Exponential decay functions have a base between ____ and ____.
4. The y-intercept of any exponential function is _____.
5. The formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ is used to calculate _____ interest.
6. The formula $A = Pe^{rt}$ is used to calculate _____ compounded interest.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so that the statement will be true. (Note: There may be more than one acceptable change.)

7. For all exponential functions $f(x) = x^b,\ b < 0$.
8. The function $f(x) = 5^x$ is an example of an exponential growth model.
9. In an exponential decay function, b^x approaches the x-axis for positive values of x.
10. The number e is defined to be approximately 3.14159.

Practice

Sketch the graph of each exponential function and label three points on each graph. (Note that some of the graphs are shifts, horizontal or vertical, of the basic exponential functions. These are similar to the shifts performed on parabolas in Chapter 16.)

1. $y = 4^x$
2. $y = 5^x$
3. $y = \left(\frac{1}{3}\right)^x$
4. $y = \left(\frac{1}{5}\right)^x$
5. $y = \left(\frac{2}{3}\right)^x$
6. $y = \left(\frac{5}{2}\right)^x$
7. $y = \left(\frac{1}{2}\right)^{-x}$
8. $y = \left(\frac{3}{4}\right)^{-x}$
9. $y = 2^{x-1}$
10. $y = 3^{x+1}$
11. $f(x) = 2^x + 1$
12. $f(x) = 3^x - 1$
13. $f(x) = -4^{-x}$
14. $g(x) = -2^{-x}$
15. $f(x) = 2^{0.5x}$
16. $g(x) = 10^{0.5x}$
17. $f(x) = 4^{-x} - 1$
18. $g(x) = 10^{-x} - 3$
19. $f(x) = 3 \cdot \left(\frac{1}{2}\right)^{x+1}$
20. $y = -4 \cdot \left(\frac{1}{3}\right)^{x-1}$

Find the following function values.

21. If $f(t) = 3 \cdot 4^t$ what is the value of $f(2)$?

22. For $f(x) = 3 \cdot 10^{2x}$, find the value of $f(0.5)$.

Use your calculator to find each value as indicated. Round your answer to the nearest hundredth.

23. Find $f(2)$ if $f(x) = 27.3 \cdot e^{-0.4x}$.

24. Find $f(3)$ if $f(x) = 41.2 \cdot e^{-0.3x}$.

25. Find $f(9)$ if $f(t) = 2000 \cdot e^{0.08t}$.

26. Find $f(22)$ if $f(t) = 2000 \cdot e^{0.05t}$.

Solve.

27. Use a graphing calculator to graph each of the following functions. In each case the x-axis is a horizontal asymptote.

a. $y = e^x$ **b.** $y = e^{-x}$ **c.** $y = e^{-x^2}$

Applications

Solve.

28. ***Bacteria:*** A biologist knows that in the laboratory, bacteria in a culture grow according to the function $y = y_0 \cdot 5^{0.2t}$, where y_0 is the initial number of bacteria present and t is time measured in hours. How many bacteria will be present in a culture at the end of 5 hours if there were 5000 present initially?

29. ***Bacteria:*** Referring to Exercise 27, how many bacteria were present initially if at the end of 15 hours, there were 2,500,000 bacteria present?

30. ***Banking:*** Four thousand dollars is deposited into a savings account with a rate of 8% per year. Find the total amount A on deposit at the end of 5 years if the interest is compounded

a. annually. **c.** quarterly. **e.** continuously.

b. semiannually. **d.** daily.

31. ***Banking:*** Find the amount A in a savings account if \$2000 is invested at 7% for 4 years and the interest is compounded

a. annually. **c.** quarterly. **e.** continuously.

b. semiannually. **d.** daily.

32. ***Investing:*** Find the value of \$1800 invested at 6% for 3 years if the interest is compounded continuously.

33. ***Investing:*** Find the value of \$2500 invested at 5% for 5 years if the interest is compounded continuously.

34. ***Sales:*** The revenue function is given by $R(x) = x \cdot p(x)$ dollars, where x is the number of units sold and $p(x)$ is the unit price. If $p(x) = 25(2)^{\frac{-x}{5}}$, find the revenue if 15 units are sold.

35. ***Sales:*** Referring to Exercise 34, if $p(x) = 40(3)^{\frac{-x}{6}}$, find the revenue if 12 units are sold.

36. ***Advertising:*** A radio station knows that during an intense advertising campaign, the number of people N who will hear a commercial is given by $N = A\left(1 - 2^{-0.05t}\right)$, where A is the number of people in the broadcasting area and t is the number of hours the commercial has been run. If there are 500,000 people in the area, how many will hear a commercial during the first 20 hours?

37. ***Investing:*** Bethany invested \$45,000 in a retirement fund that earns 8% interest and is compounded continuously, how much money will the account be worth after:

a. 10 years? **b.** 20 years? **c.** 40 years?

38. ***Technology:*** Statistics show that the fractional part of flashlight batteries f that are still good after t hours of use is given by $f = 4^{-0.02t}$. What fractional part of the batteries are still operating after 150 hours of use?

39. ***Investing:*** If a principal P is invested at a rate r compounded continuously, the interest earned is given by $I = A - P$.

a. Find the interest earned in 20 years on \$10,000 invested at 10% and compounded continuously?

b. Find the interest earned in 20 years on \$10,000 invested compounded continuously.

c. Explain why the interest earned at 5% is not just one-half of the interest earned at 10% in Parts a. and b.

40. ***Manufacturing:*** The value V of a machine at the end of t years is given by $V = C(1 - r)^t$, where C is the original cost and r is the rate of depreciation. Find the value of a machine at the end of 4 years if the original cost was \$1200 and $r = 0.20$.

41. ***Manufacturing:*** Referring to Exercise 40, find the value of a machine at the end of 3 years if the original cost was \$2000 and $r = 0.15$.

42. ***Prescriptions:*** A cancer patient is given a dose of 50 mg of a particular drug. In five days the amount of the drug in her system is reduced to 1.5625 mg. If the drug decays (or is absorbed) at an exponential rate, find the function that represents the amount of the drug. (**Hint:** Use the formula $y = y_0 b^{-t}$ and solve for b.)

43. ***Diseases:*** Determine the exponential function that fits the following information concerning exponential growth of cancer cells: $y_0 = 10{,}000$ cancer cells, and there are 160,000 cancer cells present after 4 days. (**Hint:** Use the formula $y = y_0 b^t$ and solve for b.)

Writing & Thinking

44. Discuss, in your own words, the symmetrical relationship of the graphs of the two exponential functions $y = 10^x$ and $y = 10^{-x}$.

45. Discuss, in your own words, the symmetrical relationship of the graphs of the two exponential functions $y = 10^x$ and $y = -10^x$.

Collaborative Learning

46. The following formula can be used to calculate monthly mortgage payments:

$$A = \frac{P\left(1+\frac{r}{12}\right)^n \cdot \frac{r}{12}}{\left(1+\frac{r}{12}\right)^n - 1}$$

where

A = the monthly payment,

P = amount initially borrowed (the mortgage),

r = the annual interest rate (in decimal form), and

n = the total number of monthly payments (12 times the number of years).

With the class divided into teams of 3 or 4 students, each team should complete one table (using different values for r and for P). Discuss the results as a class. Explain what this might mean for you personally.

For annual rate r = _____ and initial mortgage P = _____

Length of Mortgage (in years)	Monthly Payment A	Total Cost of Mortgage n times A
15		
20		
25		
30		

17.4 Logarithmic Functions

Objectives

A. Translate expressions between exponential and logarithmic form.

B. Evaluate logarithms by using their basic properties.

C. Use the definitions of exponential and logarithmic functions to solve equations.

D. Graph exponential and logarithmic functions on the same set of axes.

A Introduction to Logarithms

Exponential functions of the form $y = b^x$ are one-to-one functions and, therefore, have inverses. To find the inverse of a function, we interchange x and y in the equation and solve for y. Thus, for the function

$$y = b^x,$$

interchanging x and y gives the inverse function

$$x = b^y.$$

Figure 1 shows the graphs of these two functions with $b > 1$. Each function is a reflection of the other across the line $y = x$. Note that the exponential function $y = b^x$ has **the line $y = 0$ (the x-axis) as a horizontal asymptote**, and the inverse function $x = b^y$ has **the line $x = 0$ (the y-axis) as a vertical asymptote**.

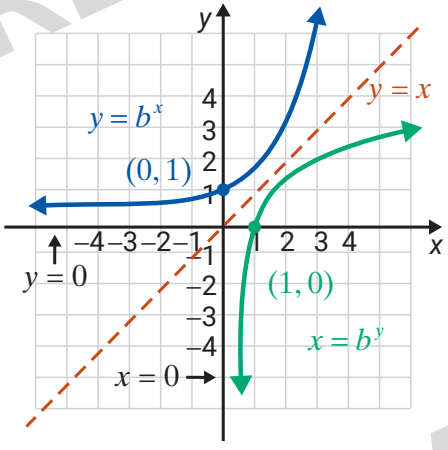


Figure 1

To solve the inverse equation $x = b^y$ for y, mathematicians have simply created a name for y. This name is logarithm (abbreviated as log). This means that **the inverse of an exponential function is a logarithmic function**.

Definition of Logarithm (base b)

For $b > 0$ and $b \neq 1$,

$$x = b^y \text{ is equivalent to } y = \log_b x.$$

$y = \log_b x$ is read "y is the logarithm (base b) of x."

DEFINITION

A logarithm is an exponent. Example 1 shows how exponential forms and logarithmic forms of equations are related.

1. Express each equation in logarithmic form.

   a. $4^1 = 4$

   b. $4^3 = 64$

   c. $4^{-1} = \frac{1}{4}$

   Express each equation in exponential form.

   d. $\log_3 9 = 2$

   e. $\log_3 81 = 4$

   f. $\log_3\left(\frac{1}{3}\right) = -1$

Example 1 Translating Between Exponential and Logarithmic Form

	Exponential Form		Logarithmic Form	
a.	$2^3 = 8$	$\Leftrightarrow$	$\log_2 8 = 3$	The base is 2. The logarithm is 3.
b.	$2^4 = 16$	$\Leftrightarrow$	$\log_2 16 = 4$	The base is 2. The logarithm is 4.
c.	$10^3 = 1000$	$\Leftrightarrow$	$\log_{10} 1000 = 3$	The base is 10. The logarithm is 3.
d.	$3^0 = 1$	$\Leftrightarrow$	$\log_3 1 = 0$	The base is 3. The logarithm is 0.
e.	$5^{-1} = \frac{1}{5}$	$\Leftrightarrow$	$\log_5 \frac{1}{5} = -1$	The base is 5. The logarithm is -1.

Note that in each case the base of the exponent is the base of the logarithm.

Now work margin exercise 1.

REMEMBER, a logarithm is an exponent. For example,

logarithm logarithm logarithm

$$10^2 = 100 \quad \text{and} \quad \log_{10} 100 = 2 \quad \text{and} \quad 100 = 10^{\log_{10} 100}$$

are all equivalent. In words,

2 is the **exponent** of the base 10 to get 100 $\left(10^2 = 100\right)$; and

2 is the **logarithm** base 10 of 100 $\left(\log_{10} 100 = 2\right)$.

B Basic Properties of Logarithms

We know that **exponents are logarithms** and from our previous knowledge of exponents, we can make the following equivalent statements for logarithms, base b.

$$b^0 = 1 \quad \Leftrightarrow \quad \log_b 1 = 0$$

$$b^1 = b \quad \Leftrightarrow \quad \log_b b = 1$$

Also, directly from the definition of $y = \log_b x$, we can make two more general statements.

$$x = b^{\log_b x} \quad \text{and} \quad \log_b b^x = x$$

In summary, we have the following four basic properties of logarithms.

Answers

1. a. $\log_4 4 = 1$

   b. $\log_4 64 = 3$

   c. $\log_4\left(\frac{1}{4}\right) = -1$

   d. $3^2 = 9$

   e. $3^4 = 81$

   f. $3^{-1} = \frac{1}{3}$

Basic Properties of Logarithms

For $b > 0$ and $b \neq 1$,

1. $\log_b 1 = 0$ — Regardless of the base, the logarithm of 1 is 0.
2. $\log_b b = 1$ — The logarithm of the base is always 1.
3. $x = b^{\log_b x}$ — for $x > 0$
4. $\log_b b^x = x$

PROPERTIES

REMEMBER, a logarithm is an exponent.

Example 2 Evaluating Logarithms

Use the four basic properties of logarithms to evaluate each expression.

a. $\log_3 1 = 0$ — By property 1

b. $\log_8 8 = 1$ — By property 2

c. $10^{\log_{10} 20} = 20$ — By property 3

d. $\log_2 32 = \log_2 2^5$ — Write 32 as 2^5 so the base is 2.

$= 5$ — By property 4

e. $\log_{10} 0.01 = \log_{10} \frac{1}{100}$

$= \log_{10} \frac{1}{10^2}$

$= \log_{10} 10^{-2}$ — Write $\frac{1}{10^2}$ as 10^{-2} so the base is 10.

$= -2$ — By property 4

Now work margin exercise 2.

2. Use the four basic properties of logarithms to evaluate each expression.

a. $\log_8 1$

b. $\log_6 6$

c. $10^{\log_{10} 30}$

d. $\log_2 64$

e. $\log_{10} 0.001$

C Solving Logarithmic Equations

Example 3 Solving Logarithmic Equations

Solve by first changing the equation to exponential form: $\log_{16} x = \frac{3}{4}$

Solution

$\log_{16} x = \frac{3}{4}$

$x = 16^{\frac{3}{4}}$ — Write the equation in exponential form and solve for x.

$x = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$

Thus, $\log_{16} 8 = \frac{3}{4}$.

Now work margin exercise 3.

3. Solve by first changing the equation to exponential form:

$\log_8 x = \frac{2}{3}$

Answers

2. a. 0 **b.** 1 **c.** 30 **d.** 6 **e.** −3

3. 4

4. Solve by first changing the equation to exponential form: $\log_4 32 = x$

Example 4 Solving Logarithmic Equations

Solve by first changing the equation to exponential form: $\log_4 8 = x$

Solution

$\log_4 8 = x$

$4^x = 8$ Write the equation in exponential form and solve for x.

$(2^2)^x = 2^3$ Use the common base, 2.

$2^{2x} = 2^3$

$2x = 3$ The exponents are equal because the bases are the same.

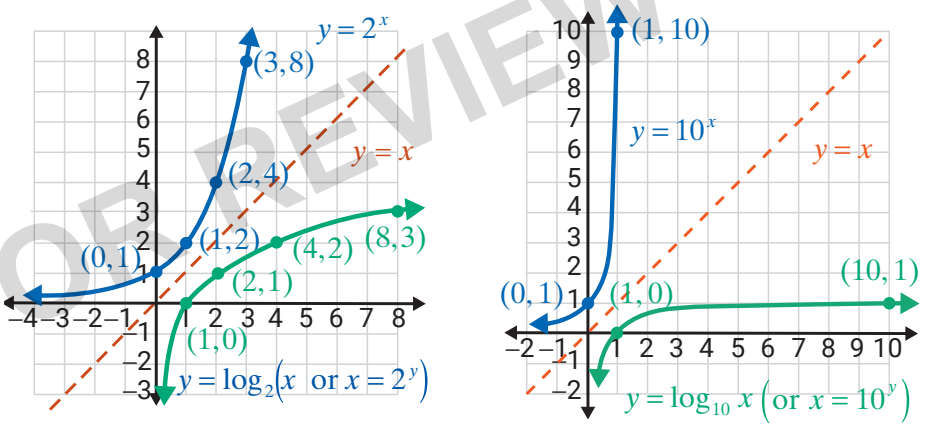
$x = \frac{3}{2}$

Thus, $\log_4 8 = \frac{3}{2}$.

Now work margin exercise 4.

D Graphs of Logarithmic Functions

As illustrated in Figure 2, because logarithmic functions are the inverses of exponential functions, the graphs of logarithmic functions can be found by reflecting the corresponding exponential functions across the line $y = x$. Figure 2a shows how the graphs of $y = 2^x$ and $y = \log_2 x$ are related. Figure 2b shows how the graphs of $y = 10^x$ and $y = \log_{10} x$ are related. Note that in the graphs of both logarithmic functions the values of y are negative when x is between 0 and 1 $(0 < x < 1)$.



a. b.

Figure 2

Recall that points on the graphs of inverse functions can be found by reversing the coordinates of ordered pairs. This means that the domain and range of a function and its inverse are interchanged. Thus, for exponential functions and logarithmic functions, we have the following.

Answers

4. $\frac{5}{2}$

- For the exponential function $y = b^x$,

  the domain is all real x, and

  the range is all $y > 0$. (The graph is above the x-axis.)

  There is a horizontal asymptote at $y = 0$.

- For the logarithmic function $y = \log_b x \left(\text{or } x = b^y\right)$,

  the domain is all $x > 0$, and (The graph is to the right of the y-axis.)

  the range is all real y.

  There is a vertical asymptote at $x = 0$.

17.4 Exercises

Concept Check

Fill-in-the-Blank. Complete each sentence using information found in this section.

1. The function $x = b^y$ is equivalent to $y =$ _____.
2. The line $y = 0$ is the _____ asymptote of $y = b^x$.
3. The inverse of an exponential function is a/an _____ function.
4. Regardless of the base, the logarithm of 1 is _____.
5. The graph of a logarithmic function can be found by _____ the corresponding exponential function across the line $y = x$.
6. The points on the graph of the inverse function can be found by _____ the coordinates of the ordered pairs.

True/False. Determine whether each statement is true or false. If a statement is false, explain how it can be changed so that the statement will be true. (Note: There may be more than one acceptable change.)

7. Exponential functions of the form $y = b^x$ are one-to-one functions and have inverses.
8. The exponent of an exponential function is the base of its inverse logarithmic function.
9. Exponents are logarithms.
10. The logarithm of the base is always 1.

Practice

Express each equation in logarithmic form. See Example 1.

1. $7^2 = 49$
2. $3^3 = 27$
3. $5^{-2} = \frac{1}{25}$
4. $2^{-5} = \frac{1}{32}$
5. $1 = \pi^0$
6. $6^0 = 1$
7. $10^2 = 100$
8. $10^1 = 10$
9. $10^k = 23$
10. $4^k = 11.6$
11. $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$
12. $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$

Express each equation in exponential form. See Example 1.

13. $\log_3 9 = 2$
14. $\log_5 125 = 3$
15. $\log_9 3 = \frac{1}{2}$
16. $\log_b 4 = \frac{2}{3}$
17. $\log_7 \frac{1}{7} = -1$
18. $\log_{1/2} 8 = -3$
19. $\log_{10} N = 1.74$
20. $\log_2 42.3 = x$
21. $\log_b 18 = 4$
22. $\log_b 39 = 10$
23. $\log_n y^2 = x$
24. $\log_b a = x^2$

Solve by first changing each equation to exponential form. See Examples 3 and 4.

25. $\log_4 x = 2$
26. $\log_3 x = 4$
27. $\log_{14} 196 = x$
28. $\log_{25} 125 = x$
29. $\log_5 \frac{1}{125} = x$
30. $\log_3 \frac{1}{9} = x$
31. $\log_{36} x = -\frac{1}{2}$
32. $\log_{81} x = -\frac{3}{4}$
33. $\log_x 32 = 5$
34. $\log_x 121 = 2$
35. $\log_8 x = \frac{5}{3}$
36. $\log_{16} x = \frac{3}{4}$
37. $\log_8 8^{3.7} = x$
38. $\log_{10} 10^{1.52} = x$
39. $\log_5 5^{\log_5 25} = x$
40. $\log_4 4^{\log_2 8} = x$

Graph each function and its inverse on the same set of axes. Label two points on each graph.

41. $f(x) = 6^x$
42. $f(x) = 2^x$
43. $y = \left(\frac{2}{3}\right)^x$
44. $y = \left(\frac{1}{4}\right)^x$
45. $f(x) = \log_4 x$
46. $f(x) = \log_5 x$
47. $y = \log_{1/2} x$
48. $y = \log_{1/3} x$
49. $y = \log_8 x$
50. $y = \log_7 x$

51. Consider the function $y = c\left(3^x\right)$ where c is a constant greater than zero. List the following:

a. The domain of the function.

b. The range of the function.

c. Any asymptotes of the graph of the function.

d. Give c two different values and sketch the graphs of both functions.

52. Consider the function $y = c\left(3^{-x}\right)$ where c is a constant greater than zero. List the following:

a. The domain of the function.

b. The range of the function.

c. Any asymptotes of the graph of the function.

d. Give c two different values and sketch the graphs of both functions.

Writing & Thinking

53. Discuss, in your own words, the symmetrical relationship of the graphs of the two functions $y = 10^x$ and $y = \log_{10} x$.

54. Discuss, in your own words, the symmetrical relationship of the graphs of the two logarithmic functions $y = \log_{10} x$ and $y = -\log_{10} x$.

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