3 *Scalar and Vector Fields*

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3.1 Scalar Fields: Isotimic Surfaces: Gradients

If to each point (x,y,z) of a region in space there is made to correspond a number $f(x,y,z)$, we say that f is a *scalar field.* In other words, a scalar field is simply a scalar-valued function of three variables.

Here are some physical examples of scalar fields: the mass density of the atmosphere, the temperature at each point in an insulated wall, the water pressure at each point in the ocean, the gravitational potential of points in astronomical space, the electrostatic potential of the region between two condenser plates. Such scalar fields as density and pressure are only approximate idealizations of a complicated physical situation, since they take no account of the atomic properties of matter.

For the sake of fixing ideas, the following scalar fields are given as examples that will be referred to repeatedly:

1. $f(x,y,z) = x + 2y - 3z$
2. $f(x,y,z) = x^2 + y^2 + z^2$
3. $f(x,y,z) = x^2 + y^2$
4. $f(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2$
5. $f(x,y,z) = \sqrt{x^2 + y^2} - z$
6. $f(x,y,z) = \dfrac{1}{x^2 + y^2}$

The fields in examples 1 through 5 are defined at every point in space. The field in example 6 is defined at every point (x,y,z) except where $x^2 + y^2 = 0$, that is, everywhere except on the z axis.

If f is a scalar field, any surface defined by $f(x,y,z) = C$, where C is a constant, is called an *isotimic surface* (from the Greek *isotimos,* meaning "of equal value"). Sometimes, in physics, more specialized terms are used. For instance, if f denotes either electric or gravitational field potential, such surfaces are called *equipotential surfaces.* If f denotes temperature, they are called *isothermal surfaces.* If f denotes pressure, they are called *isobaric surfaces.*

In the above examples, the isotimic surfaces are (see fig. 3.1):

1. all planes perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
2. all spheres with center at the origin
3. all right circular cylinders with the z axis as axis of symmetry
4. a family of ellipsoids
5. a family of cones
6. the same as in example 3

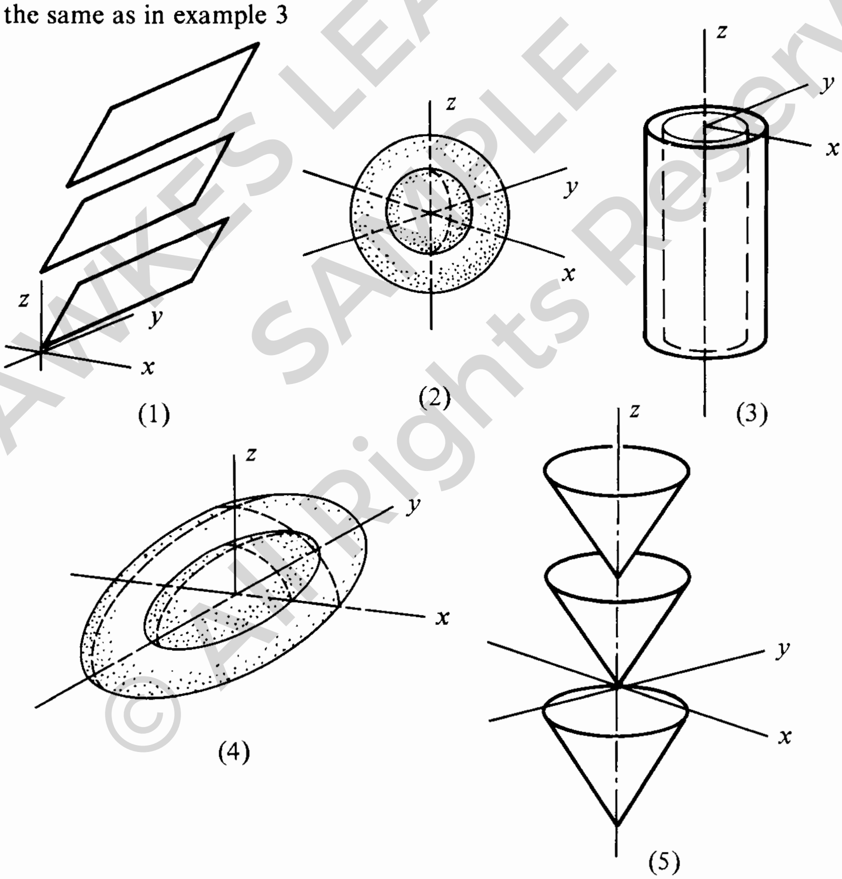

Figure 3.1

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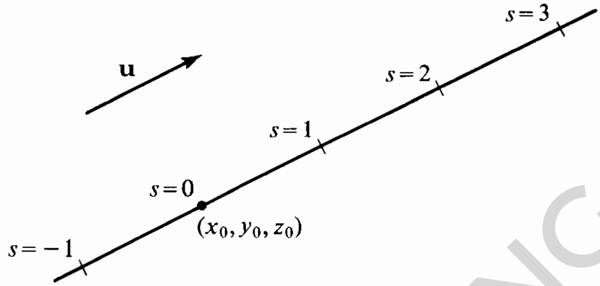

Figure 3.2

It is impossible for distinct isotimic surfaces of the same scalar field to intersect, since only one number $f(x,y,z)$ is associated with any one point (x,y,z).

Let us consider the behavior of a scalar field in the neighborhood of a point (x_0,y_0,z_0) within its region of definition. We imagine a line segment passing through (x_0,y_0,z_0) parallel to a given vector **u.** Let s denote the displacement measured along the line segment in the direction of **u** (fig. 3.2), with $s = 0$ corresponding to (x_0,y_0,z_0). To each value of the parameter s there corresponds a point (x,y,z) on the line segment, and hence a corresponding scalar $f(x,y,z)$. The derivative df/ds at $s = 0$, if this derivative exists, is called the *directional derivative* of f at (x_0,y_0,z_0), in the direction of the vector **u.**

In other words, the directional derivative of f is simply the rate of change of f, per unit distance, in some prescribed direction. The directional derivative df/ds will generally depend on the location of the point (x_0,y_0,z_0) and also on the direction prescribed.

The directional derivative of a scalar field f in a direction parallel to the x axis, with s measured as increasing in the positive x direction, is conventionally denoted $\partial f/\partial x$, and is called the partial derivative of f with respect to x. Similarly, the directional derivative of f in the positive y direction is called $\partial f/\partial y$, and that in the positive z direction, $\partial f/\partial z$. We assume that the reader has had some experience with partial derivatives.

The directional derivative of a scalar field f in a direction that is not parallel to any of the coordinate axes is conventionally denoted df/ds, but of course this symbol is ambiguous; it would not make sense to ask "what is df/ds" without specifying the direction in which s is to be measured.

A convenient way of specifying the desired direction is by prescribing a vector **u** pointing in that direction. Although the magnitude of **u** is immaterial, it is conventional to take **u** to be a unit vector. We have already seen (section 2.2) that a unit vector in a desired direction can be obtained by computing $d\mathbf{R}/ds$ in that direction, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. That is,

$$\mathbf{u} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{3.1}$$

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is a unit vector pointing in the direction in which s is measured. Here we are thinking of x, y, and z as functions of the parameter s, for points (x,y,z) on the line segment; s is, of course, arc length along the segment.

If the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous throughout a region, then it is well known (see Appendix B for a proof) that the following chain rule is valid:

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{3.2}$$

If we define the *gradient* of f to be the vector

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{3.3}$$

we see that the right side of (3.2) is the dot product of $\mathbf{u}$ with $\mathbf{grad}\, f$:

$$\frac{df}{ds} = \mathbf{u} \cdot \mathbf{grad}\, f \tag{3.2$'$}$$

Since $\mathbf{u}$ is a unit vector, $\mathbf{u} \cdot \mathbf{grad}\, f = |\mathbf{u}||\mathbf{grad}\, f| \cos\theta = |\mathbf{grad}\, f| \cos\theta$, where θ is the angle between $\mathbf{grad}\, f$ and $\mathbf{u}$. This gives us the first fundamental property of the gradient.

PROPERTY 3.1 *The component of* **grad** *f in any given direction gives the directional derivative df/ds in that direction.*

By the maximum principle (example 1.16), the largest possible value of $\mathbf{u} \cdot \mathbf{grad}\, f$, for unit vectors $\mathbf{u}$, is obtained when $\mathbf{u}$ is in the same direction as $\mathbf{grad}\, f$ (assuming that $\mathbf{grad}\, f \neq \mathbf{0}$). Since $\mathbf{u} \cdot \mathbf{grad}\, f = df/ds$, it follows that the maximum value of df/ds is obtained in the direction of $\mathbf{grad}\, f$. This is the second fundamental property of the gradient.

PROPERTY 3.2 **grad** *f points in the direction of the maximum rate of increase of the function f.*

If $\mathbf{u}$ points in the direction of $\mathbf{grad}\, f$, then

$$\mathbf{u} \cdot \mathbf{grad}\, f = |\mathbf{u}||\mathbf{grad}\, f| \cos\theta = |\mathbf{grad}\, f|$$

which gives the third fundamental property of the gradient.

PROPERTY 3.3 *The magnitude of* **grad** *f equals the maximum rate of increase of f per unit distance.*

Experience has shown that the wording of these fundamental properties makes them rather easy to memorize [and they *should* be memorized, together with the definition (3.3)].

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The fourth fundamental property of the gradient of a function makes it possible to use the gradient concept in solving geometrical problems.

PROPERTY 3.4 *Through any point* (x_0,y_0,z_0) *where* $\mathbf{grad}\, f \neq \mathbf{0}$, *there passes an isotimic surface* $f(x,y,z) = C$; $\mathbf{grad}\, f$ *is normal (i.e., perpendicular) to this surface at the point* (x_0,y_0,z_0).

This property holds only when $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ exist and are continuous in a neighborhood of the point in question. The constant C is, of course, equal to $f(x_0,y_0,z_0)$. If $\mathbf{grad}\, f = \mathbf{0}$, the locus of points satisfying $f(x,y,z) = C$ might not form a surface. (Consider, for example, this locus if f is a *constant* function.)

We omit a detailed proof of this fourth property, but the following discussion may make it seem reasonable. Let C denote the value of f at (x_0,y_0,z_0). Since $\mathbf{grad}\, f \neq \mathbf{0}$, it follows from the preceding fundamental properties that df/ds will be positive in some directions. If, then, we proceed away from (x_0,y_0,z_0) in one of these directions, the value of $f(x,y,z)$ will increase, and if we proceed in the opposite direction, its value will decrease. Since f and its partial derivatives are continuous, it seems reasonable that there will be a surface passing through (x_0,y_0,z_0) on which f remains at the constant value C; on one side of this surface the values of f will be greater than (and on the other, less than) C. Now suppose we consider any smooth arc passing through (x_0,y_0,z_0) and entirely contained in this surface. Then $f(x,y,z) = C$ for all points on this arc, and so $df/ds = 0$, where s is measured along this arc. Since $df/ds = \mathbf{u} \cdot \mathbf{grad}\, f$, and in this case $\mathbf{u}$ is a unit vector tangent to this arc, we see that $\mathbf{u} \cdot \mathbf{grad}\, f = df/ds = 0$, implying that $\mathbf{grad}\, f$ is perpendicular to $\mathbf{u}$. This reasoning applies to any smooth arc in the surface passing through (x_0,y_0,z_0). Hence $\mathbf{grad}\, f$ is perpendicular to every such arc, at that point, which can be the case only if $\mathbf{grad}\, f$ is perpendicular to the surface (fig. 3.3).

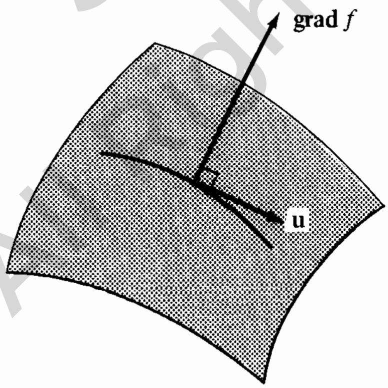

Figure 3.3

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We now return to the six examples given previously. In each case the gradient is easily computed using the definition (3.3):

1. $\mathbf{grad}\, f = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
2. $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
3. $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j}$
4. $\mathbf{grad}\, f = \dfrac{x}{2}\mathbf{i} + \dfrac{2y}{9}\mathbf{j} + 2z\mathbf{k}$
5. $\mathbf{grad}\, f = \dfrac{x\mathbf{i} + y\mathbf{i}}{\sqrt{x^2 + y^2}} - \mathbf{k}$
6. $\mathbf{grad}\, f = \dfrac{-2x}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2y}{(x^2 + y^2)^2}\mathbf{j}$

1. (This is the only one of the six examples for which $\mathbf{grad}\, f$ is a constant.) We already know from section 1.10 that $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ is perpendicular to any plane of the form $x + 2y - 3z = C$. We see that $\mathbf{grad}\, f = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Thus we have verified the fourth fundamental property in this special case.
2. In this case the isotimic surfaces are spheres centered at the origin, so the normals to these surfaces must be vectors pointing directly away from the origin. Sure enough, we have $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{R}$, and we know that the vector $2\mathbf{R}$ always points directly away from the origin. To see the significance of the number 2 here, let R denote the distance from the origin to the point (x,y,z). Then we can, in this example, write the function in terms of R: it is simply R^2. Moreover, if we move away from any point in the direction of maximum increase of R^2, which obviously means moving directly away from the origin, then the element of arc length is simply dR. In this direction, the derivative df/ds is df/dR, and $(d/dR)(R^2) = 2R$. Also, $|2\mathbf{R}| = 2R$, so we have verified the third fundamental property in this special case.
3. The reader familiar with cylindrical coordinates can do the same thing here as we just did with example 2. Let $\rho = (x^2 + y^2)^{1/2}$, the distance from the point (x,y,z) to the z axis. The function f in this example is simply ρ^2 and obviously increases most rapidly in a direction perpendicular to the z axis. Its derivative in this direction is 2ρ, which is also the magnitude $|\mathbf{grad}\, f| = (4x^2 + 4y^2)^{1/2}$. The direction is clearly normal to the isotimic surfaces, since the latter are right circular cylinders centered on the z axis. The second, third, and fourth fundamental properties are transparent in this case, as they were in example 2.
5. [We skip example 4.] All we care to note here is the elementary geometrical significance of the $-\mathbf{k}$ term in $\mathbf{grad}\, f$. The isotimic surfaces of this function are conical; each has an apex on the z axis and spreads outward with increasing z. Thus, we see easily that the normal to one such surface will not point directly away from the z axis, as it does in example 3, but will have an additional, constant component in the negative z direction.

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The following examples are some sample problems that illustrate the use of the fundamental properties of the gradient of a scalar field.

Example 3.1 Find df/ds in the direction of the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ at the point $(1,1,2)$ if $f(x,y,z) = x^2 + y^2 - z$.

Solution $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at $(1,1,2)$. A unit vector in the desired direction is $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (obtained by dividing $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ by its own length). Property 3.1 then gives $df/ds = \mathbf{u} \cdot \mathbf{grad}\, f = \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = 3$. This means that the value of the function f is increasing 3 units per unit distance if we proceed from $(1,1,2)$ in the direction stated.

Example 3.2 The temperature of points in space is given by $f(x,y,z) = x^2 + y^2 - z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that he will get cool as soon as possible. In what direction should he move?

Solution As we saw in example 3.1, $\mathbf{grad}\, f = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ at $(1,1,2)$. The mosquito should move in the direction $-\mathbf{grad}\, f$, since $\mathbf{grad}\, f$ is in the direction of increasing temperature.

Example 3.3 A mosquito is flying at a speed of 5 units of distance per second, in the direction of the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. The temperature is given by $f(x,y,z) = x^2 + y^2 - z$. What is his rate of increase of temperature, per unit time, at the instant he passes through the point $(1,1,2)$?

Solution As shown in example 3.1, df/ds in this direction is 3 units per unit distance. The rate of increase of temperature per unit time is thus $df/dt = (df/ds)(ds/dt) = (3)(5) = 15$ degrees per second.

Example 3.4 What is the maximum possible df/ds, if $f(x,y,z) = x^2 + y^2 - z$, at the point $(1,4,2)$?

Solution $|\mathbf{grad}\, f| = |2\mathbf{i} + 8\mathbf{j} - \mathbf{k}| = \sqrt{69}$. The answer is approximately 8.31 units per unit distance.

Example 3.5 Find a unit vector normal to the surface $x^2 + y^2 - z = 6$ at the point $(2,3,7)$.

Solution This is an isotimic surface for the function $f(x,y,z) = x^2 + y^2 - z$. At $(2,3,7)$ we have $\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} = 4\mathbf{i} + 6\mathbf{j} - \mathbf{k}$. The length of this vector is $\sqrt{53}$. Thus an answer is $(\sqrt{53}/53)(4\mathbf{i} + 6\mathbf{j} - \mathbf{k})$. (The negative of this vector is also a correct answer.)

The reader may have observed that the number 6, the constant on the right-hand side of the equation defining the isotimic surface in example 3.5, appears to have no effect on the normal, $\mathbf{grad}\, f$. This is not quite true. Granted, the formula

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for **grad** f ignores the 6, but when it is evaluated at (x,y,z), the numbers x, y, and z must satisfy $x^2 + y^2 - z = 6$. Clearly $2^2 + 3^2 - 7 = 6$.

EXERCISES

1. Compute **grad** f if
   (a) $f = \sin x + e^{xy} + z$
   (b) $f = 1/|\mathbf{R}|$
   (c) $f = \mathbf{R} \cdot \mathbf{i} \times \mathbf{j}$
2. If $f(x,y,z) = x^2 + y^2$, what is the locus of points in space for which **grad** f is parallel to the y axis?
3. What can you say about a function whose gradient is everywhere parallel to the y axis?
4. Find all functions $f(x,y,z)$ such that **grad** $f = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$.
5. Can you find a scalar whose gradient is $y\mathbf{i}$?
6. Describe **grad** f in words, without actually doing any calculating, given that $f(x,y,z)$ is the distance between (x,y,z) and the z axis.
7. Given $f(x,y,z) = x^2 + y^2 + z^2$, find the maximum value of df/ds at the point $(3,0,4)$,
   (a) by using the gradient of f.
   (b) by interpreting f geometrically.
8. A volcano just erupted and lava is streaming down from the mountain top. Suppose that the altitude of the mountain is given by

$$z(x,y) = he^{-(x^2 + 2y^2)}$$

   where h is the maximum height, and suppose also that lava flows in the direction of steepest descent (fastest change in z). Find
   (a) the projection on the xy plane of the direction in which lava flows away from the point $(1,2,he^{-9})$.
   (b) the equation of the projection on the xy plane of the flow line of the lava passing through the point $(1,2,he^{-9})$.
9. Find the derivative of $f(x,y,z) = x + xyz$ at the point $(1,-2,2)$ in the direction of
   (a) $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
   (b) $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
10. Find the directional derivative df/ds at $(1,3,-2)$ in the direction of $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ if
    (a) $f(x,y,z) = yz + xy + xz$
    (b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$
    (c) $f(x,y,z) = xy + x^3y^3$
    (d) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$
11. Find the magnitude of the greatest rate of change of $f(x,y,z) = (x^2 + z^2)^3$ at $(1,3,-2)$. Interpret geometrically.

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12. Find the direction of maximal increase of the function $f(x,y,z) = e^{-xy} \cos z$ at the point $(1,1,0)$.

13. By vector methods, find the point on the curve $x = t$, $y = t^2$, $z = 2$ at which the temperature $\phi(x,y,z) = x^2 - 6x + y^2$ takes its minimum value.

14. Find a vector normal to the surface $x^2 + yz = 5$ at $(2,1,1)$.

15. Find an equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 21$ at $(2,4,-1)$.

16. Find a vector normal to the cylinder $x^2 + z^2 = 8$ at $(2,0,2)$,
(a) by inspection (draw a diagram).
(b) by finding the gradient of the function $f(x,y,z) = x^2 + z^2$ at $(2,0,2)$.

17. Find an equation of the plane tangent to the surface $z^2 - xy = 14$ at $(2,1,4)$.

18. Find equations of the line normal to the sphere $x^2 + y^2 + z^2 = 2$ at $(1,1,0)$,
(a) by inspection (draw a diagram).
(b) by computing the gradient of $f(x,y,z) = x^2 + y^2 + z^2$ at $(1,1,0)$, and using this to find the normal.

19. Find a unit vector normal to the plane $3x - y + 2z = 3$,
(a) by the methods of section 1.10.
(b) by the methods of the preceding section.

20. Find an equation of the plane tangent to the surface $z = x^2 + y^2$ at $(2,3,13)$. [*Hint:* Consider the function $f(x,y,z) = x^2 + y^2 - z$.]

21. Let $T(x,y,z) = x^2 + 2y^2 + 3z^2$, and let S be the isotimic surface: $T = 1$. Find all points (x,y,z) on S that have tangent planes with normals $(1,1,1)$.

22. If $\phi(x,y,z) = x^2y + zy + z^3$, find
(a) the gradient of ϕ.
(b) the equation of the plane passing through the point $(1,-1,1)$ and tangent to the level surface of ϕ at that point.

23. Find a unit vector tangent to the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$ at the point $(\sqrt{2},\sqrt{2},\sqrt{5})$,
(a) by drawing a diagram, obtaining the answer by inspection.
(b) by finding the vector product of the normals to the two surfaces at that point.
(c) by writing the equation of the curve in parametric form. (*Hint:* Let $x = 2 \sin t$ and $y = 2 \cos t$.)

24. Determine the angle between the normals of the intersecting spheres $x^2 + y^2 + z^2 = 16$ and $(x - 1)^2 + y^2 + z^2 = 16$ at the point $(1/2, 3/2, 3\sqrt{6}/2)$.

25. At what angle does the line $2x = y = 2z$ intersect the ellipsoid $2x^2 + y^2 + 2z^2 = 8$?

26. What is the angle between the tangent to the curve

$$\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t^2\mathbf{k} \qquad (0 \le t \le 3)$$

and the normal to the surface $z = 16 - x^2 - y$ at their point of intersection?

27. At what angle does the curve $x = t$, $y = 2t - t^2$, $z = 2t^4$ intersect the surface $x^2 + y^3 + 3z^2 = 14$ at the point $(1,1,2)$?

28. Find the angle between the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + (z - 3)^2 = 9$ at the point $(2,-1,5)$.

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29. Let S_1 and S_2 be the surfaces with equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$$

Show that, if $a^2B^2 - b^2A^2 = 0$, then the curve of intersection of S_1 and S_2 must be parallel to the xy plane.

30. If $\mathbf{R}_1$ denotes the position vector of a point P relative to an origin O_1 in the xy plane, and $\mathbf{R}_2$ denotes the position vector of the same point relative to another origin O_2, then $|\mathbf{R}_1| + |\mathbf{R}_2| =$ constant is the equation of an ellipse with foci O_1 and O_2. Use this observation to prove that lines O_1P and O_2P make equal angles with the tangent to the ellipse at P. [*Hint:* **grad** $(|\mathbf{R}_1| + |\mathbf{R}_2|)$ is normal to the ellipse.]

31. Find the point on the sphere $x^2 + y^2 + z^2 = 84$ that is nearest the plane $x + 2y + 4z = 77$.

32. Find the point on the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ that is nearest the plane $x + 2y + 3z = 8$.

33. What point on the curve $x = t$, $y = t^2$, $z = 2$ is closest to the surface $x^2 - 6x + y^2 + 7 = 0$?

34. Show that any level curve $\mathbf{R}(t)$ for the function $f(x,y,z)$ satisfies

$$\frac{d\mathbf{R}}{dt} \cdot \mathbf{grad}\, f = 0$$

35. Given $\phi = \tan^{-1} x + \tan^{-1} y$ and $\psi = (x + y)/(1 - xy)$, show that $\nabla\phi \times \nabla\psi = 0$. [*Hint:* It is easy if you recognize the formula for $\tan(A + B)$.]

36. Given $w = uv$, where u and v are scalar fields, show that $\nabla w \cdot \nabla u \times \nabla v = 0$,
(a) by direct calculation.
(b) without calculation.

37. Generalize the result of the preceding exercise.

3.2 Vector Fields and Flow Lines

A *vector field* **F** is a rule associating with each point (x,y,z) in a region a vector $\mathbf{F}(x,y,z)$. In other words, a vector field is a vector-valued function of three variables.

In visualizing a vector field, we imagine that from each point in the region there extends a vector. Both direction and magnitude may vary with position (fig. 3.4). A good visualization of a vector field is a sandstorm, with $\mathbf{F}(\mathbf{R})$ giving the velocity of the sand particle at **R**.

Some vector fields are not defined for all points in space. For example, the vector field

$$\mathbf{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$$

is not defined along the z axis, since $x^2 + y^2 = 0$ for points on the z axis.

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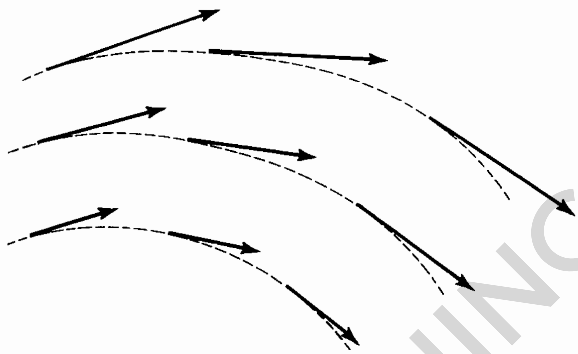

Figure 3.4

A vector field may be written in terms of its components:

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$

Example 3.6 If $f(x,y,z)$ is a *scalar* field, **grad** f is a *vector* field.

Example 3.7 Each of the "vectors" $\mathbf{u}_r$ and $\mathbf{u}_\theta$ (section 2.5) is a vector field defined in the plane.

Example 3.8 In hydrodynamics, one associates with each point of a region the velocity of the fluid passing that point. In this manner one obtains, at any instant of time, a vector field describing the instantaneous velocity of the fluid at every point.

Example 3.9 In theoretical physics, there is associated with each point in space an electric field intensity vector, representing the force that would be exerted, per unit charge, on a charged particle, if it were located at that point. This electric field, at any instant of time, constitutes a vector field. (Magnetic fields and gravitational fields also provide examples of vector fields defined in space.)

Let us consider a vector field **F** that is defined and nonzero at every point of a region in space. Any curve passing through the region is called a *flow line* of **F** provided that, at every point on the curve, **F** is tangent to the curve. (Flow lines are also called *stream lines* or *characteristic curves* of **F**. If **F** is a force field, the flow lines are commonly called *lines of force*.) In figure 3.4, three flow lines are indicated as dotted curves.

This may be visualized another way. The vector field **F** determines, at each point in the region, a direction. If a particle moves in such a manner that the direction of its velocity at any point coincides with the direction of the vector field **F** at that point, the space curve traced out is a flow line.

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If the vector field $\mathbf{F}(x,y,z)$ describes the velocity at each point in a hydrodynamic system, the flow lines are the paths traversed by the component particles of the fluid, assuming that $\mathbf{F}$ is not a function of time. (The situation is more complicated for time-varying flows.)

Note that if $g(x,y,z)$ is a scalar field that is not zero at any point, the flow lines of the vector field $g(x,y,z)\mathbf{F}(x,y,z)$ will be the same as those of $\mathbf{F}(x,y,z)$, since only the *direction* of $\mathbf{F}$ at any point is relevant in determining the flow lines.

Since the direction of a flow line is uniquely determined by the field $\mathbf{F}$, it is impossible to have two different flow directions at the same point, and therefore it is impossible for two flow lines to cross. If the magnitude of $\mathbf{F}$ is zero at some point in space, then no direction is defined at that point and no flow line passes through that point. Now let's see how to calculate flow lines.

If $\mathbf{R}$ is the position vector to an arbitrary point of a flow line, and if s represents arc length measured along the flow line, then the unit vector tangent to the curve at that point is given by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \tag{3.4}$$

The requirement that $\mathbf{T}$ have the same direction as $\mathbf{F}$ can be written

$$\mathbf{T} = \beta\mathbf{F} \tag{3.5}$$

where β is a scalar-valued function of x, y, and z. This can be written in terms of components:

$$\beta F_1 = \frac{dx}{ds} \qquad \beta F_2 = \frac{dy}{ds} \qquad \beta F_3 = \frac{dz}{ds} \tag{3.6}$$

If F_1, F_2, and F_3 are all nonzero, we may eliminate β and write eq. (3.6) in differential form:

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \tag{3.7}$$

If one of these functions (say F_3) is identically zero in a region, then we obtain directly from eq. (3.6) that the curve lies in a plane (say, z = constant) parallel to one of the coordinate planes.

Example 3.10 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, then $F_1 = x$, $F_2 = y$, and $F_3 = 1$, giving $dx/x = dy/y = dz$. Solving the differential equations $dx/x = dz$ and $dy/y = dz$, we obtain $x = C_1e^z$, $y = C_2e^z$. Thus the equations of the flow line passing through the point (3,4,7) are $x = 3e^{z-7}$, $y = 4e^{z-7}$. The equations of the flow line passing through the origin are $x = 0$, $y = 0$—i.e., the z axis.

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Example 3.11 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, then $F_1 = x$, $F_2 = y$, and $F_3 = 0$. In this case eq. (3.6) becomes $\beta x = dx/ds$, $\beta y = dy/ds$, and $0 = dz/ds$. Eliminating β from the first two equations, we obtain $dx/x = dy/y$, and, solving, we obtain $y = Cx$. From the third equation we obtain $z =$ constant. The field is zero when both x and y equal zero, and so the flow lines are not defined along the z axis. The flow lines are straight half-lines parallel to the xy plane, extending outward from the z axis.

Example 3.12 If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, then $-\beta y = dx/ds$, $\beta x = dy/ds$, and $0 = dz/ds$. Thus $-dx/y = dy/x$, and hence $x^2 + y^2 =$ constant. Also, we have $z =$ constant. The flow lines are circles surrounding the z axis and are parallel to the xy plane. As in example 3.11, no flow lines pass through points on the z axis.

Flow lines may be infinite in extent, as in examples 3.10 and 3.11, or they may close upon themselves, as in example 3.12.

EXERCISES

1. For the vector field $\mathbf{F}$ of example 3.12, draw a diagram similar to figure 3.4 showing the values of F at the points (1,0), (0,1), (−1,0), (0,−1), (1,1), (−1,1), (−1,−1), (1,−1), and a scattering of other points. Indicate flow lines.
2. Let $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$.
   (a) Find the general equation of a flow line.
   (b) Find the flow line through the point (1,1,2).
3. Without doing any calculating, describe the flow lines of the vector field $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [*Hint:* If a particle located at (x,y,z) has velocity $\mathbf{R}$, in what direction is it moving relative to the origin?]
4. The flow lines of the gradient of a scalar field cross the isotimic surfaces orthogonally. Explain.

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3.3 Divergence

The concept of *gradient,* as we have presented it, describes the rate of change of a scalar field. We now consider the more complicated problem of describing the rate of change of a *vector* field. There are two fundamental measures of this rate of change: the *divergence* and the *curl.*

Roughly speaking, the divergence of a vector field is a scalar field that tells us, at each point, the extent to which the field *explodes,* or diverges, from that point. The curl of a vector field is a vector field that gives us, at each point, an indication of how the field swirls in the vicinity of that point (fig. 3.5). However, to describe divergence and curl in such a brief manner is both useless and a bit dangerous, since

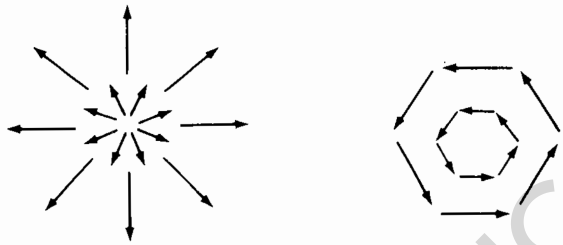

Figure 3.5

(if taken literally) both of these preceding sentences are not only vague but technically incorrect. As we shall see, it is possible for a field to have a positive divergence without appearing to "diverge" at all, and it is possible for a field to have a nontrivial curl and yet have flow lines that do not bend at all.

In this section we consider only the divergence. We begin by presenting a heuristic derivation, which will serve to motivate the formal definition. As usual the vector field will be denoted by

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

Let us once again picture the velocity field for a stream of flowing particles, such as sand particles in a sandstorm, electrons in a wire or plasma, or fluid particles in a jet emerging from a nozzle. Denote the number of particles per unit volume—the *particle density*—by ν. Now if each particle weighs m grams (say), then the *mass density* μ equals $m\nu$ grams per unit volume.

Next let $\mathbf{v}(x,y,z)$ be the velocity of the fluid particle located at (x,y,z); $\mathbf{v}$ is the velocity field of the fluid. The vector field

$$\mathbf{F}(x,y,z) = \mu(x,y,z)\mathbf{v}(x,y,z)$$

is called the *mass flow rate density* of the fluid. We shall use F to calculate the mass flow rate, or number of grams per unit time, that crosses any hypothetical "window" in the flow pattern.

Thus consider a small planar patch of surface area δS inside the fluid, as in figure 3.6; the arrows depict the velocity field $\mathbf{v}(x,y,z)$. If we start counting particles crossing δS for the next Δt seconds, which will be the last particles to make it through? Clearly they are the ones that are $-(\mathbf{v}\Delta t)$ away from the patch at the start—and as figure 3.6(b) illustrates, all the particles in the cylinder with base ΔS and slant height $|\mathbf{v}|\Delta t$ cross ΔS in that time.

How many particles are in this cylinder? Its *volume* is given by

$$(\text{base area}) \text{ times } (\text{height}) = \Delta S\, \mathbf{n} \cdot \mathbf{v}\Delta t$$

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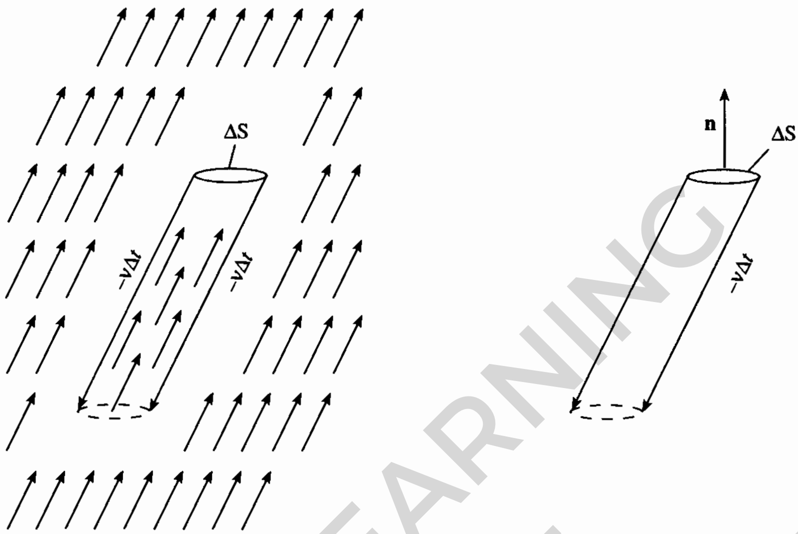

(a) (b)

Figure 3.6

where **n** is the unit normal to the patch as shown; therefore it contains $\nu(\Delta S)\mathbf{n} \cdot \mathbf{v}\Delta t$ particles. As a result the mass flow rate through the small area ΔS *per unit time* is given by

$$[m\nu\mathbf{v}] \cdot \mathbf{n}\Delta S = \mu\mathbf{v} \cdot \mathbf{n}\Delta S = \mathbf{F} \cdot \mathbf{n}\Delta S$$

This is called the *flux of the vector field* **F** *through* ΔS.

[If we had used the charge q per particle instead of the mass m and defined the vector field $\mathbf{j}(x,y,z)$ in terms of the charge density $\rho = q\nu$

$$\mathbf{j}(x,y,z) = \rho(x,y,z)\mathbf{v}(x,y,z)$$

then the flux of the "current density" **j** through ΔS would give the charge flow rate, or *current*, through the patch.]

To define the divergence of the field **F**, we imagine an infinitesimal rectangular parallelepiped having corners at (x,y,z), $(x + \Delta x, y, z)$, $(x, y + \Delta y, z)$, $(x, y, z + \Delta z)$, and so on (fig. 3.7). We shall compute the total flux of the field **F** through the six sides of this box in the outward direction (i.e., on each side we choose **n** to be the outward normal). We then divide this flux by the volume of the box and take the limit as the dimensions of the box go to zero. *This limit is called the divergence of* **F** *at the point* (x,y,z). In other words, the divergence is the net outflux per unit volume.

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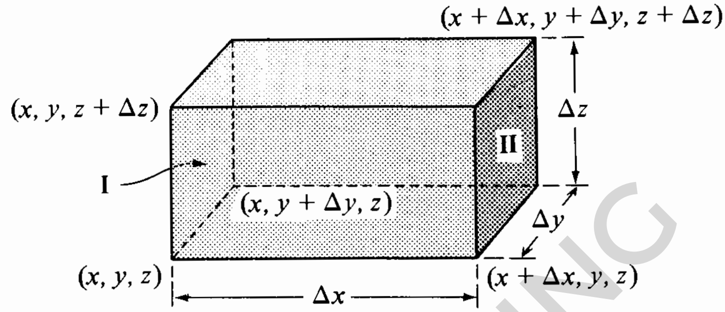

Figure 3.7

The computation of this limit proceeds as follows. On face I in figure 3.7 the outward normal is $-\mathbf{i}$. Thus, according to the above analysis, the flux out of this face is approximately $-F_1\Delta y\Delta z$. The flux out of face II, whose outward normal is $\mathbf{i}$, is $F_1\Delta y\Delta z$. These need not cancel, however, since F_1 may grow or diminish along the length Δx of the box. To account for this we estimate the total flux out of faces I and II as

$$[F_1(x + \Delta x, y, z) - F_1(x,y,z)]\Delta y\Delta z$$

The difference in these values of F_1 is given, to the same order of accuracy, by

$$\frac{\partial F_1}{\partial x}\Delta x$$

Thus the contribution to the net outward flux from faces I and II is

$$\frac{\partial F_1}{\partial x}\Delta x\Delta y\Delta z$$

Similarly, the two faces in the y direction contribute

$$\frac{\partial F_2}{\partial y}\Delta y\Delta x\Delta z$$

and, adding the contribution of the two remaining faces, we see that the net outward flux is approximately

$$\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\Delta x\Delta y\Delta z$$

After we divide by the volume $\Delta x\Delta y\Delta z$, our approximations become accurate as we take the limit, and we are led to the following statement, which we take as our formal *definition* of divergence:

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The divergence of a vector field

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

is a scalar field, denoted div **F**, *defined by*

$$\text{div } \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{3.8}$$

It is easy to compute the divergence of a vector field, as we demonstrate with examples. Keep in mind that div **F** *is defined* by eq. (3.8), and that our heuristic discussion gives us the interpretation of div **F** as net outflux per unit volume.

Example 3.13 Find div **F** if $\mathbf{F} = x\,\mathbf{i} + y^2z\,\mathbf{j} + xz^3\,\mathbf{k}$.

Solution

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(xz^3)\\ &= 1 + 2yz + 3xz^2\end{aligned}$$

Example 3.14 Find div **F** if $\mathbf{F} = xe^y\,\mathbf{i} + e^{xy}\,\mathbf{j} + \sin yz\,\mathbf{k}$.

Solution

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(e^{xy}) + \frac{\partial}{\partial z}(\sin yz)\\ &= e^y + xe^{xy} + y\cos yz\end{aligned}$$

Example 3.15 Give an example of a vector field **F** that has divergence equal to 3 at every point in space.

Solution Many solutions can be given, for instance $\mathbf{F} = 3x\mathbf{i}$ or $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Example 3.16 In figure 3.8, is the divergence of **F** at point P positive or negative? Assume no variation of **F** in the z direction and that F_3 is identically zero.

Solution Heuristically, we can see from the diagram that the flux through the x faces of a parallelepiped at P will cancel, while there is definitely flux *out* of both y faces. Since there is no flux in the z direction, we expect that the divergence is positive.

More rigorously, we observe that F_1 is approximately constant, so $\partial F_1/\partial x = 0$. Below P, F_2 is negative, and above P, F_2 is positive, so $\partial F_2/\partial y$ is positive. Since $F_3 = 0$, we have $\partial F_3/\partial z = 0$. It follows that div **F** is positive at point P.

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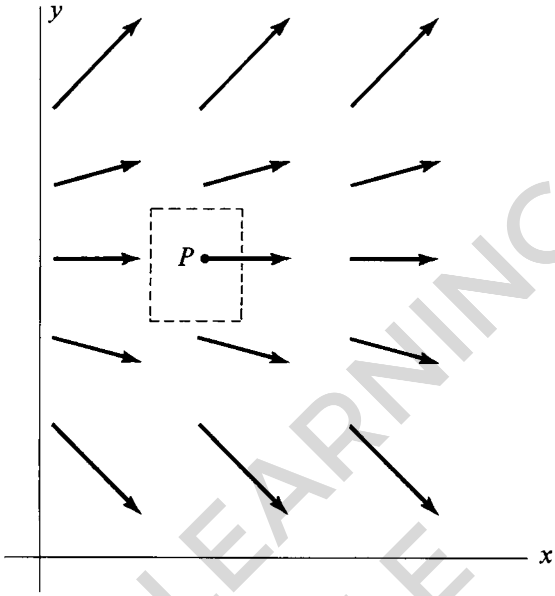

Figure 3.8

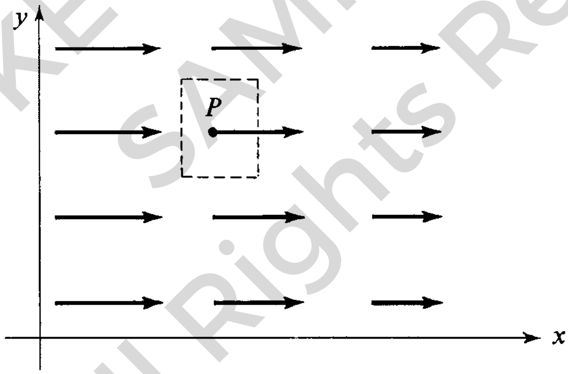

Figure 3.9

Example 3.17 In figure 3.9, is the divergence of **F** at point P positive or negative? Assume no variation of **F** in the z direction and that F_3 is identically zero.

Solution Again heuristically, there is no flux in the y or z direction, and the flux in the x direction *decreases* as we move to the right. So the net flux through the sides of a box at P is inward, and the divergence must be negative.

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More precisely, we note that F_1 is decreasing with increasing x; hence $\partial F_1/\partial x$ is negative. F_2 and F_3 are zero at every point. It follows that the divergence of **F** is negative at every point.

In figure 3.8 where the divergence is positive, the lines of flux do, in a sense, diverge in a neighborhood of P. This is the picture that motivates the common (incorrect) statement that "positive divergence means the field is diverging, negative divergence means the field is converging." Note that in figure 3.9 the divergence is negative, but the flow lines are not converging. The divergence is negative because more fluid enters a given region from the left than leaves it to the right.

Returning to our mental image of **F** as a mass flow rate density $\mu\mathbf{v}$, we can derive an important relationship by considering the consequences of the principle of conservation of mass. The quantity

$$\operatorname{div}(\mu\mathbf{v})\ \Delta x\Delta y\Delta z$$

measures the flux of $(\mu\mathbf{v})$ out of a box with dimensions Δx, Δy, and Δz. But we have seen that the flux measures the mass of fluid crossing the faces of the box. Therefore this outflux must result in a decrease in the amount of fluid in the box

$$\mu\Delta x\Delta y\Delta z$$

and hence a decrease in the density. Thus we can write

$$\operatorname{div}(\mu\mathbf{v}) = -\frac{\partial\mu}{\partial t} \tag{3.9}$$

This is called the *equation of continuity* in fluid mechanics. The corresponding equation for charge density

$$\operatorname{div}(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t}$$

expresses the conservation of charge.

The heuristic reasoning employed in this section is, of course, subject to criticism, as are most arguments involving "infinitesimals." Its rigorous justification rests on a result known, appropriately enough, as the *divergence theorem*, and we will study it in the next chapter. For the present, we are satisfied with having a formal, precise definition of div **F** in eq. (3.8), and an intuitive picture of what it represents.

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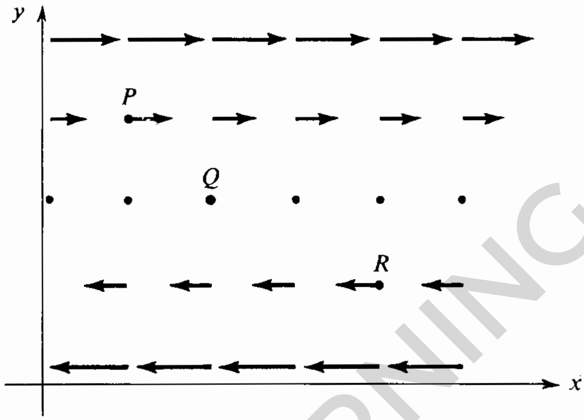

Figure 3.10

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EXERCISES

1. Find div **F**, given that $\mathbf{F} = e^{xy}\,\mathbf{i} + \sin xy\,\mathbf{j} + \cos^2 zx\,\mathbf{k}$.
2. Find div **F**, given that $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
3. Find div **F**, given that $\mathbf{F} = \mathbf{grad}\,\phi$, where $\phi = 3x^2y^3z$.
4. Find the divergence of the field

$$\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

   Is the divergence of this field defined at every point in space?
5. Show in detail that $\operatorname{div}(\phi\mathbf{F}) = \phi \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \mathbf{grad}\,\phi$.
6. Construct an example of a scalar field ϕ and a vector field **F**, neither of which is constant, for which $\operatorname{div}(\phi\mathbf{F})$ is identically equal to $\phi \operatorname{div} \mathbf{F}$.
7. Give an example of a nonconstant field with zero divergence.
8. Give an example of a field with a constant negative divergence.
9. Give an example of a field whose divergence depends only on x, is always positive, and increases with increasing x. (*Hint:* The function e^x is positive for every x.)
10. What can you say about the divergence of the vector field in figure 3.10 at points P, Q, and R? Assume no variation of **F** in the z direction and that F_3 is identically zero.
11. What can you say about the divergence of the vector field in figure 3.11 at points P, Q, and R? Assume no variation of **F** in the z direction and that F_3 is identically zero.
12. Another hydrodynamic interpretation of divergence is as follows. Let **F** be the velocity field of a fluid. Consider a small rectangular parallelepiped of fluid located at (x,y,z). Then the divergence of **F** is the time rate of change of volume of this body of fluid, per

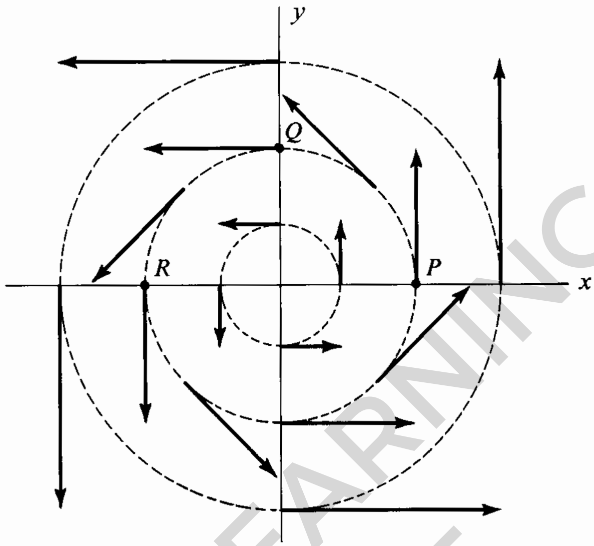

Figure 3.11

unit volume, as the size of the box goes to zero. Show this. [*Hint:* With $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the box initially has corners at $\mathbf{R}$, $\mathbf{R} + \Delta x\mathbf{i}$, $\mathbf{R} + \Delta y\mathbf{j}$, $\mathbf{R} + \Delta z\mathbf{k}$, etc. After time Δt these corners have moved to the new positions $\mathbf{R} + \mathbf{F}(x,y,z)\Delta t$, $\mathbf{R} + \Delta x\mathbf{i} + \mathbf{F}(x + \Delta x, y, z)\Delta t$, $\mathbf{R} + \Delta y\mathbf{j} + \mathbf{F}(x, y + \Delta y, z)\Delta t$, $\mathbf{R} + \Delta z\mathbf{k} + \mathbf{F}(x, y, z + \Delta z)\Delta t$, etc. Calculate the new volume using the triple scalar product, and compute the limit described above.]

3.4 Curl

As in the previous section, we shall preface our formal definition of the curl of a vector field with some heuristic considerations. Once again imagine a flowing liquid with velocity field $\mathbf{v}(x,y,z)$ and *unit* density $\mu = 1$; thus the mass flow rate density is $\mathbf{F} = \mu\mathbf{v}$ is simply $\mathbf{v}$ itself. Consider a small paddle wheel, like that shown in figure 3.12, that is free to rotate about its axis AA'.

Imagine that we immerse this paddle wheel in the liquid. Because of the flow of the liquid, it will tend to rotate with some angular velocity. This angular velocity will vary, depending on where we locate the paddle wheel and on the positioning of its axis. For definiteness we shall compute the angular velocity with the paddle wheel lined up along the z axis.

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