ALGEBRA

Properties of Absolute Value

For all real numbers a and b:

$|a| \geq 0$ $\quad$ $|-a| = |a|$

$a \leq |a|$ $\quad$ $|ab| = |a||b|$

$\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \; b \neq 0$

$|a+b| \leq |a| + |b|$ $\quad$ Triangle Inequality

Properties of Integer Exponents and Radicals

Assume that n and m are positive integers, that a and b are nonnegative, and that all denominators are nonzero. See Appendices B and D for graphs and further discussion.

$a^n \cdot a^m = a^{n+m}$ $\quad$ $(a^n)^m = a^{nm}$

$\dfrac{a^n}{a^m} = a^{n-m}$ $\quad$ $(ab)^n = a^n b^n$

$a^{-n} = \dfrac{1}{a^n}$ $\quad$ $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

$a^{1/n} = \sqrt[n]{a}$ $\quad$ $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\quad$ $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Special Product Formulas

$(A-B)(A+B) = A^2 - B^2$

$(A+B)^2 = A^2 + 2AB + B^2$

$(A-B)^2 = A^2 - 2AB + B^2$

$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Factoring Special Binomials

$A^2 - B^2 = (A-B)(A+B)$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Distance Formula

The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Slope of a Line

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Horizontal lines $y = c$ have slope 0.

Vertical lines $x = c$ have undefined slope.

Parallel and Perpendicular Lines

Given a line with slope m:

slope of parallel line $= m$

slope of perpendicular line $= -1/m$

Forms of Equations of a Line

Standard Form: $ax + by = c$

Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y-intercept

Point-Slope Form: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line

Properties of Logarithms

Let a, b, x, and y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number. See Appendix B for graphs and further discussion.

$\log_a x = y$ and $x = a^y$ are equivalent

$\log_a 1 = 0$ $\quad$ $\log_a a = 1$

$\log_a (a^x) = x$ $\quad$ $a^{\log_a x} = x$

$\log_a (xy) = \log_a x + \log_a y$

$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

$\log_a (x^r) = r \log_a x$

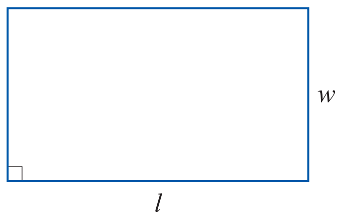
$\log_b x = \dfrac{\log_a x}{\log_a b}$ $\quad$ Change of base formula

GEOMETRY

A = area, C = circumference, SA = surface area or lateral area, V = volume

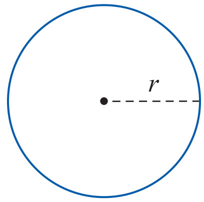
Rectangle

$$A = lw$$



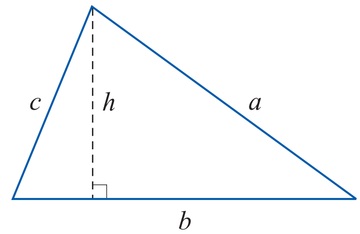
Circle

$$A = \pi r^2 \qquad C = 2\pi r$$



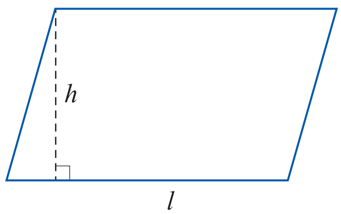
Triangle

$$A = \frac{1}{2}bh$$



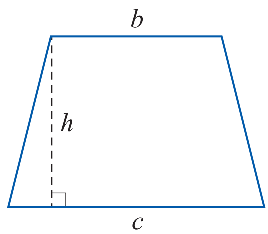
Parallelogram

$$A = lh$$



Trapezoid

$$A = \frac{1}{2}h(b+c)$$



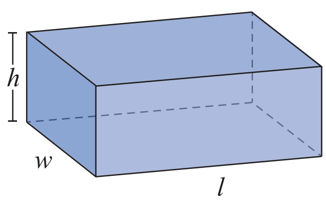
Heron's Formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$

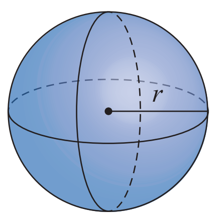
Rectangular Prism

$$V = lwh \qquad SA = 2lh + 2wh + 2lw$$



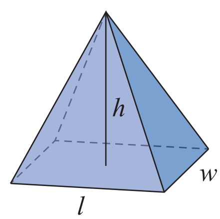
Sphere

$$V = \frac{4}{3}\pi r^3 \qquad SA = 4\pi r^2$$



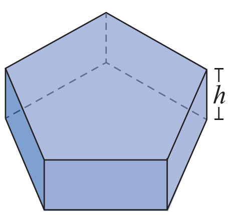
Rectangular Pyramid

$$V = \frac{1}{3}lwh$$



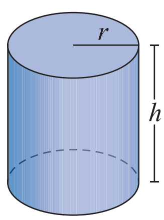
Right Cylinder

$$V = (\text{Area of Base})h$$



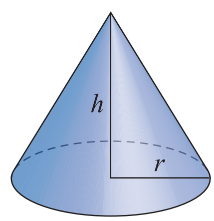
Right Circular Cylinder

$$V = \pi r^2 h \qquad SA = 2\pi r^2 + 2\pi rh$$



Cone

$$V = \frac{1}{3}\pi r^2 h \qquad SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

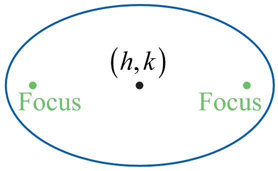


Trigonometric and Hyperbolic Functions: Definitions, Graphs, and Identities

See Appendix C.

CONIC SECTIONS

Ellipse



Let $a, b > 0$ with $a \geq b$.
Center: (h,k)
Major axis length: $2a$
Minor axis length: $2b$
Standard form of equation:

1. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

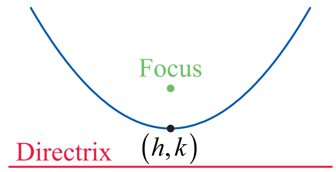
major axis is horizontal

2. $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

major axis is vertical

Foci: on major axis, c units away from the center, where $c^2 = a^2 - b^2$

Parabola



Let $p \neq 0$.
Vertex: (h,k)
Standard form of equation:

1. $(x-h)^2 = 4p(y-k)$

vertically oriented

Focus: $(h, k+p)$

Directrix: $y = k - p$

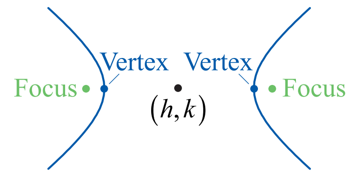
2. $(y-k)^2 = 4p(x-h)$

horizontally oriented

Focus: $(h+p, k)$

Directrix: $x = h - p$

Hyperbola



Let $a, b > 0$.
Center: (h,k)
Standard form of equation:

1. $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

foci are aligned horizontally

Asymptotes: $y - k = \pm\dfrac{b}{a}(x-h)$

2. $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

foci are aligned vertically

Asymptotes: $y - k = \pm\dfrac{a}{b}(x-h)$

Foci: c units away from the center, where $c^2 = a^2 + b^2$

Vertices: a units away from the center

LIMITS

Definition of Limit

Let f be a function defined on an open interval containing c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is L, and write $\lim_{x\to c} f(x) = L$, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever x satisfies $0 < |x - c| < \delta$.

Basic Limit Laws

Sum Law:

$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

Difference Law:

$$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

Constant Multiple Law:

$$\lim_{x\to c}[kf(x)] = k\lim_{x\to c} f(x)$$

Product Law:

$$\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

Quotient Law:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \text{ provided } \lim_{x\to c} g(x) \neq 0$$

The Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at c itself, and if $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$ as well.

Continuity at a Point

Given a function f defined on an open interval containing c, we say f is continuous at c if

$$\lim_{x\to c} f(x) = f(c).$$

L'Hôpital's Rule

Suppose f and g are differentiable at all points of an open interval I containing c, and that $g'(x) \neq 0$ for all $x \in I$ except possibly at $x = c$. Suppose further that either

$$\lim_{x\to c} f(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x) = 0$$

or

$$\lim_{x\to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to c} g(x) = \pm\infty.$$

Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

assuming the limit on the right is a real number or ∞ or $-\infty$.

DERIVATIVES

The Derivative of a Function

The derivative of f, denoted f', is the function whose value at the point x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{\ln a}\cdot\frac{1}{x}$$

Elementary Differentiation Rules

Constant Rule:

$$\frac{d}{dx}(k) = 0$$

Constant Multiple Rule:

$$\frac{d}{dx}\left[kf(x)\right] = k\frac{d}{dx}f(x)$$

Sum Rule:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference Rule:

$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Power Rule:

$$\frac{d}{dx}(x^r) = rx^{r-1}$$

Chain Rule:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{x^2-1}},\quad x>1$$

$$\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2},\quad |x|<1$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=\frac{-1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}},\quad 0<x<1$$

$$\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{1-x^2},\quad |x|>1$$

The Derivative Rule for Inverse Functions

If a function f is differentiable on an interval (a,b), and if $f'(x)\neq 0$ for all $x\in(a,b)$, then f^{-1} both exists and is differentiable on the image of the interval (a,b) under f, denoted as $f((a,b))$ in the formula below. Further,

$$\text{if } x\in(a,b), \text{ then } \left(f^{-1}\right)'(f(x))=\frac{1}{f'(x)},$$

and

$$\text{if } x\in f((a,b)), \text{ then } \left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}.$$

The Mean Value Theorem

If f is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there is at least one point $c\in(a,b)$ for which

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

INTEGRATION

Properties of the Definite Integral

Given the integrable functions f and g on the interval $[a,b]$ and any constant k, the following properties hold.

1. $\int_a^a f(x)\,dx=0$
2. $\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx$
3. $\int_a^b k\,dx=k(b-a)$
4. $\int_a^b kf(x)\,dx=k\int_a^b f(x)\,dx$
5. $\int_a^b [f(x)\pm g(x)]\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx$
6. $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx$, assuming each integral exists
7. If $f(x)\le g(x)$ on $[a,b]$, then
$$\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx.$$
8. If $m=\min_{a\le x\le b} f(x)$ and $M=\max_{a\le x\le b} f(x)$, then
$$m(b-a)\le\int_a^b f(x)\,dx\le M(b-a).$$

The Fundamental Theorem of Calculus

Part I

Given a continuous function f on an interval I and a fixed point $a\in I$, define the function F on I by $F(x)=\int_a^x f(t)\,dt$. Then $F'(x)=f(x)$ for all $x\in I$.

Part II

If f is a continuous function on the interval $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=F(b)-F(a).$$

The Substitution Rule

If $u=g(x)$ is a differentiable function whose range is the interval I, and if f is continuous on I, then

$$\int f(g(x))g'(x)\,dx=\int f(u)\,du.$$

Hence, if F is an antiderivative of f on I,

$$\int f(g(x))g'(x)\,dx=F(g(x))+C.$$

Integration by Parts

Given differentiable functions f and g,

$$\int f(x)g'(x)\,dx=f(x)g(x)-\int g(x)f'(x)\,dx.$$

If we let $u=f(x)$ and $v=g(x)$, then $du=f'(x)\,dx$ and $dv=g'(x)\,dx$ and the equation takes on the more easily remembered differential form

$$\int u\,dv=uv-\int v\,du.$$

SEQUENCES AND SERIES

Summation Facts and Formulas

Constant Rule for Finite Sums:

$$\sum_{i=1}^{n} c = nc, \text{ for any constant } c$$

Constant Multiple Rule for Finite Sums:

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \text{ for any constant } c$$

Sum/Difference Rule for Finite Sums:

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sum of the First *n* Positive Integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Sum of the First *n* Squares:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of the First *n* Cubes:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Series

For a geometric sequence $\{a_n\}$ with common ratio r:

Partial Sum:

$$s_n = \frac{a(1-r^n)}{1-r}, \text{ if } r \neq 0, 1$$

Infinite Sum:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \text{ if } |r| < 1$$

Binomial Series

For any real number m and $-1 < x < 1$,

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$$
$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots$$

where

$$\binom{m}{0} = 1, \quad \binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and $\displaystyle\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$ for $n \geq 3$

Taylor Series and Maclaurin Series

Given a function f with derivatives of all orders throughout an open interval containing a, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series generated by f about a. The Taylor series generated by f about 0 is also known as the Maclaurin series generated by f.