ALGEBRA

Properties of Absolute Value

For all real numbers a and b:

$|a| \geq 0$ $\qquad$ $|-a| = |a|$

$a \leq |a|$ $\qquad$ $|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|},\ b \neq 0$ $\qquad$ $|a+b| \leq |a| + |b|$ $\quad$ Triangle Inequality

Properties of Integer Exponents and Radicals

Assume that n and m are positive integers, that a and b are nonnegative, and that all denominators are nonzero. See Appendices B and D for graphs and further discussion.

$a^n \cdot a^m = a^{n+m}$ $\qquad$ $(a^n)^m = a^{nm}$

$\frac{a^n}{a^m} = a^{n-m}$ $\qquad$ $(ab)^n = a^n b^n$

$a^{-n} = \frac{1}{a^n}$ $\qquad$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^{1/n} = \sqrt[n]{a}$ $\qquad$ $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\qquad$ $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Special Product Formulas

$(A-B)(A+B) = A^2 - B^2$

$(A+B)^2 = A^2 + 2AB + B^2$

$(A-B)^2 = A^2 - 2AB + B^2$

$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Factoring Special Binomials

$A^2 - B^2 = (A-B)(A+B)$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Distance Formula

The distance d between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Slope of a Line

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Horizontal lines $y = c$ have slope 0.

Vertical lines $x = c$ have undefined slope.

Parallel and Perpendicular Lines

Given a line with slope m:

slope of parallel line $= m$

slope of perpendicular line $= -1/m$

Forms of Equations of a Line

Standard Form: $ax + by = c$

Slope-Intercept Form: $y = mx + b$, where m is the slope and b is the y-intercept

Point-Slope Form: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line

Vector Form: $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$, where $\mathbf{r}_0$ is a fixed vector and $\mathbf{v}$ is a direction vector

Properties of Logarithms

Let a, b, x, and y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number. See Appendix B for graphs and further discussion.

$\log_a x = y$ and $x = a^y$ are equivalent

$\log_a 1 = 0$ $\qquad$ $\log_a a = 1$

$\log_a (a^x) = x$ $\qquad$ $a^{\log_a x} = x$

$\log_a (xy) = \log_a x + \log_a y$

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a (x^r) = r \log_a x$

$\log_b x = \frac{\log_a x}{\log_a b}$ $\quad$ Change of base formula

Trigonometric and Hyperbolic Functions: Definitions, Graphs, and Identities

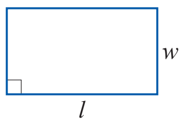
See Appendix C.

GEOMETRY

$A =$ area, $C =$ circumference, $SA =$ surface area or lateral area, $V =$ volume

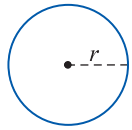
Rectangle

$A = lw$



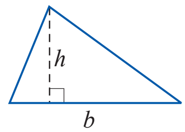
Circle

$A = \pi r^2 \quad C = 2\pi r$



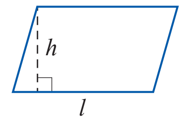
Triangle

$A = \frac{1}{2}bh$



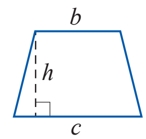
Parallelogram

$A = lh$



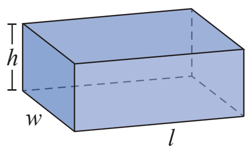
Trapezoid

$A = \frac{1}{2}h(b + c)$



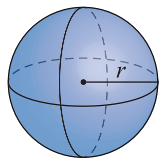
Rectangular Prism

$V = lwh \quad SA = 2lh + 2wh + 2lw$



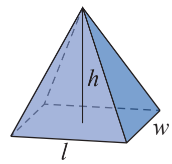
Sphere

$V = \frac{4}{3}\pi r^3 \quad SA = 4\pi r^2$



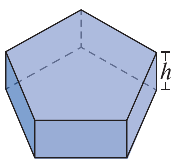
Rectangular Pyramid

$V = \frac{1}{3}lwh$



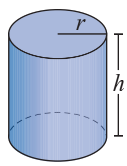
Right Cylinder

$V = (\text{Area of Base})h$



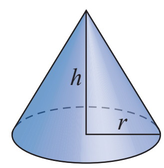
Right Circular Cylinder

$V = \pi r^2 h \quad SA = 2\pi r^2 + 2\pi rh$



Cone

$V = \frac{1}{3}\pi r^2 h \quad SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$



LIMITS

Definition of Limit

Let f be a function defined on an open interval containing c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is L, and write $\lim\limits_{x \to c} f(x) = L$, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever x satisfies $0 < |x - c| < \delta$.

Basic Limit Laws

Sum/Difference Law: $\lim\limits_{x \to c}\left[f(x) \pm g(x)\right] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x)$

Constant Multiple Law: $\lim\limits_{x \to c}\left[kf(x)\right] = k\lim\limits_{x \to c} f(x)$

Product Law: $\lim\limits_{x \to c}\left[f(x)g(x)\right] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

Quotient Law: $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, provided $\lim\limits_{x \to c} g(x) \neq 0$

Squeeze Theorem

If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at c itself, and if $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$, then $\lim\limits_{x \to c} f(x) = L$ as well.

Continuity at a Point

Given a function f defined on an open interval containing c, we say f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

L'Hôpital's Rule

Suppose f and g are differentiable at all points of an open interval I containing c, and that $g'(x) \neq 0$ for all $x \in I$ except possibly at $x = c$. Suppose further that either

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0$$

or

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty.$$

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right is a real number or ∞ or $-\infty$.

DERIVATIVES

The Derivative of a Function

The derivative of f, denoted f', is the function whose value at the point x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Elementary Differentiation Rules

Constant Rule: $\frac{d}{dx}(k) = 0$

Constant Multiple Rule: $\frac{d}{dx}\left[kf(x)\right] = kf'(x)$

Sum/Difference Rule: $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$

Power Rule: $\frac{d}{dx}(x^r) = rx^{r-1}$

Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad |x| < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, \quad |x| > 1$$

The Derivative Rule for Inverse Functions

If a function f is differentiable on an interval (a,b), and if $f'(x) \neq 0$ for all $x \in (a,b)$, then f^{-1} both exists and is differentiable on the image of the interval (a,b) under f, denoted as $f((a,b))$ in the formula below. Further,

$$\text{if } x \in (a,b), \text{ then } \left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)},$$

and

$$\text{if } x \in f((a,b)), \text{ then } \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

The Mean Value Theorem

If f is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there is at least one point $c \in (a,b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

INTEGRATION

Properties of the Definite Integral

Given the integrable functions f and g on the interval $[a,b]$ and any constant k, the following properties hold.

1. $\int_a^a f(x)dx = 0$

2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

3. $\int_a^b k\,dx = k(b-a)$

4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

6. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$, assuming each integral exists

7. If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

8. If $m = \min_{a\le x\le b} f(x)$ and $M = \max_{a\le x\le b} f(x)$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is the interval I, and if f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Hence, if F is an antiderivative of f on I,

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Integration by Parts

Given differentiable functions f and g,

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

If we let $u = f(x)$ and $v = g(x)$, then $du = f'(x)dx$ and $dv = g'(x)dx$ and the equation takes on the more easily remembered differential form

$$\int u\,dv = uv - \int v\,du.$$

The Fundamental Theorem of Calculus

Part I

Given a continuous function f on an interval I and a fixed point $a \in I$, define the function F on I by $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$ for all $x \in I$.

Part II

If f is a continuous function on the interval $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

SEQUENCES AND SERIES

Summation Facts and Formulas

Constant Rule for Finite Sums:

$$\sum_{i=1}^{n} c = nc, \text{ for any constant } c$$

Constant Multiple Rule for Finite Sums:

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \text{ for any constant } c$$

Sum/Difference Rule for Finite Sums:

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sum of the First *n* Positive Integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Sum of the First *n* Squares:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of the First *n* Cubes:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Series

For a geometric sequence $\{a_n\}$ with common ratio r:

Partial Sum: $s_n = \dfrac{a(1-r^n)}{1-r}$, if $r \ne 0, 1$

Infinite Sum: $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$, if $|r| < 1$

Binomial Series

For any real number m and $-1 < x < 1$:

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n$$

$$= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$+ \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots$$

Taylor Series and Maclaurin Series

Given a function f with derivatives of all orders throughout an open interval containing a, the power series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

is called the Taylor series generated by f about a. The Taylor series generated by f about 0 is also known as the Maclaurin series generated by f.

VECTOR CALCULUS

Properties of Scalar Multiplication and Vector Addition

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and scalars a and b:

Scalar Multiplication Properties	Vector Addition Properties
$a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
$(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$	$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$
$(ab)\mathbf{u} = a(b\mathbf{u}) = b(a\mathbf{u})$	$\mathbf{u}+\mathbf{0}=\mathbf{u}$
$1\mathbf{u}=\mathbf{u}$, $0\mathbf{u}=\mathbf{0}$, and $a\mathbf{0}=\mathbf{0}$	$\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
$\lvert a\mathbf{u}\rvert = \lvert a\rvert\lvert\mathbf{u}\rvert$	

Dot Product

Given two vectors $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$, the dot product $\mathbf{u}\cdot\mathbf{v}$ of the two vectors is the scalar defined by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3.$$

A similar formula defines the dot product of two vectors in $\mathbb{R}^2$.

Properties of the Dot Product

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and scalar a:

$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

$\mathbf{0}\cdot\mathbf{u}=0$

$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

$a(\mathbf{u}\cdot\mathbf{v}) = (a\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(a\mathbf{v})$

$\mathbf{u}\cdot\mathbf{u} = \lvert\mathbf{u}\rvert^2$

Dot Product and the Angle between Two Vectors

If two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are depicted so that their initial points coincide, and if θ represents the smaller of the two angles formed by $\mathbf{u}$ and $\mathbf{v}$ (so that $0\le\theta\le\pi$), then

$$\mathbf{u}\cdot\mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta.$$

Projection of u onto v

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}.$$

Cross Product

Given $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$,

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$= \begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\mathbf{i} - \begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\mathbf{j} + \begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\mathbf{k}$$

$$= (u_2v_3-u_3v_2)\mathbf{i} - (u_1v_3-u_3v_1)\mathbf{j} + (u_1v_2-u_2v_1)\mathbf{k}.$$

Properties of the Cross Product

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ and scalars a and b:

$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$

$\mathbf{0}\times\mathbf{u}=\mathbf{0}$

$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$

$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$

$(a\mathbf{u})\times(b\mathbf{v}) = (ab)(\mathbf{u}\times\mathbf{v})$

$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Coordinate Conversion Relationships

Cylindrical and Cartesian	Spherical and Cylindrical	Spherical and Cartesian
$r^2=x^2+y^2$	$\rho^2=r^2+z^2$	$\rho^2=x^2+y^2+z^2$
$x=r\cos\theta$	$r=\rho\sin\varphi$	$x=\underbrace{\rho\sin\varphi}_{r}\cos\theta$
$y=r\sin\theta$	$\theta=\theta$	$y=\underbrace{\rho\sin\varphi}_{r}\sin\theta$
$z=z$	$z=\rho\cos\varphi$	$z=\rho\cos\varphi$

Gradient Vector

Given a function $f(x_1,x_2,\ldots,x_n)$,

$$\nabla f(x_1,x_2,\ldots,x_n) = \left\langle f_{x_1}(x_1,x_2,\ldots,x_n), f_{x_2}(x_1,x_2,\ldots,x_n),\ldots,f_{x_n}(x_1,x_2,\ldots,x_n)\right\rangle.$$

Computation of $D_{\mathbf{u}} f(c)$ (Directional Derivative)

Assuming the derivative of $f(x,y)$ at the point (a,b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ exists,

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle = \nabla f(a,b) \cdot \mathbf{u}.$$

More generally, if $f(x_1, x_2, \ldots, x_n)$ is differentiable at the point $c = (c_1, c_2, \ldots, c_n)$ and if $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ is a unit vector, then

$$D_{\mathbf{u}} f(c) = \nabla f(c) \cdot \mathbf{u}.$$

Properties of the Gradient

Assume f and g are both differentiable functions and that k is a fixed real number. Then the following laws hold.

Sum/Difference Law: $\nabla(f \pm g) = \nabla f \pm \nabla g$

Constant Multiple Law: $\nabla(kf) = k\nabla f$

Product Law: $\nabla(fg) = f\nabla g + g\nabla f$

Quotient Law: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, provided $g \neq 0$

The Fundamental Theorem for Line Integrals (Gradient Theorem)

Assume that f is a differentiable function whose gradient ∇f is continuous along a curve C and that C is defined by the smooth vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Divergence (Flux Density) of a Vector Field

The divergence, or flux density, of a vector field $\mathbf{F}(x,y,z) = \langle P, Q, R \rangle$ is the scalar function

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

In general, we can denote the divergence of a vector field $\mathbf{F}$ as the dot product of the del operator and $\mathbf{F}$.

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Curl of a Vector Field

The curl of a vector field

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

is the vector function

$$\text{curl } \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

The curl of a vector field $\mathbf{F}$ can be remembered as the cross product of the del operator and $\mathbf{F}$.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

Green's Theorem (Tangential-Curl Form)

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let R be the region enclosed by C. If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and P and Q have continuous partial derivatives on an open region containing R, then

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{T}\, ds &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_C P\,dx + Q\,dy \\ &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.\end{aligned}$$

Extending $\mathbf{F}$ to $\mathbf{F}(x,y,z) = \langle P(x,y), Q(x,y), 0 \rangle$ and using the fact that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k} \cdot \mathbf{k} = (\text{curl } \mathbf{F}) \cdot \mathbf{k} = \nabla \times \mathbf{F} \cdot \mathbf{k},$$

we can write this version of the formula as

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA.$$

Green's Theorem (Normal-Divergence Form)

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let R be the region enclosed by C. If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ and P and Q have continuous partial derivatives on an open region containing R, then

$$\begin{aligned}\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \oint_C P\,dy - Q\,dx \\ &= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \\ &= \iint_R \nabla \cdot \mathbf{F}\, dA.\end{aligned}$$

Stokes' Theorem

Assume $\mathbf{F}$ is a vector field with continuous partial derivatives in an open region of space containing a piecewise smooth surface S. Assume that the boundary of S is a simple, closed, piecewise smooth curve C, and that C is positively oriented with respect to S. Then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

The Divergence Theorem

Assume $\mathbf{F}$ is a vector field with continuous partial derivatives in an open region of space containing $D \subseteq \mathbb{R}^3$, and assume D is either a simple union or a finite union of simple regions. Let S denote the surface of D, and let $\mathbf{n}$ be the outward-pointing field of unit vectors normal to S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$