6.3 The Slope-Intercept Form

Objectives

A Interpret the slope of a line as a rate of change.

B Calculate the slope of a line given two points that lie on the line.

C Find the slopes of and graph horizontal and vertical lines.

D Recognize the slope-intercept form for a linear equation in two variables: $y = mx + b$.

Objective A The Meaning of Slope

If you ride a bicycle up a mountain road, you certainly know when the **slope** (a measure of steepness called the **grade** for roads) increases because you have to pedal harder. The contractor who built the road was aware of the **slope** because trucks traveling the road must be able to control their downhill speed and be able to stop in a safe manner. A carpenter given a set of house plans calling for a roof with a **pitch** of 7 : 12 knows that for every 7 feet of rise (vertical distance) there are 12 feet of run (horizontal distance). That is, the ratio of rise to run is $\frac{rise}{run} = \frac{7}{12}$.

Figure 1

Note that this ratio can be in units other than feet, such as inches or meters. (See Figure 2.)

$$\frac{rise}{run} = \frac{7 \text{ inches}}{12 \text{ inches}} = \frac{3.5 \text{ feet}}{6 \text{ feet}} = \frac{14 \text{ feet}}{24 \text{ feet}}$$

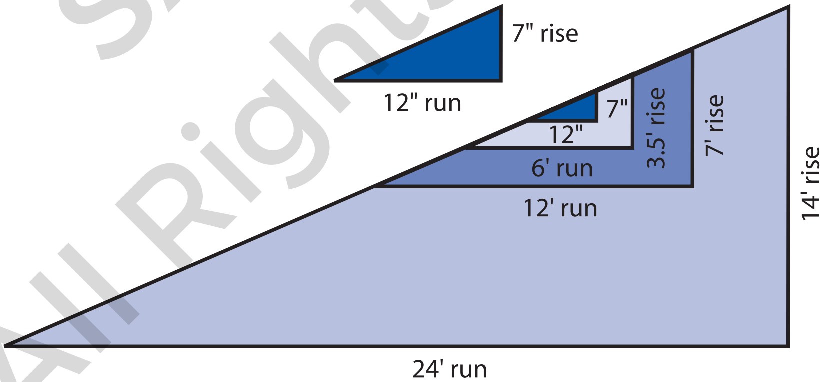


Figure 2

For a line, the **ratio of rise to run** is called the **slope of the line**. The graph of the linear equation $y = \frac{1}{3}x + 2$ is shown in Figure 3. What do you think is the slope of the line? Do you think that the slope is positive or negative? Do you think the slope might be $\frac{1}{3}$? $\frac{3}{1}$? 2?

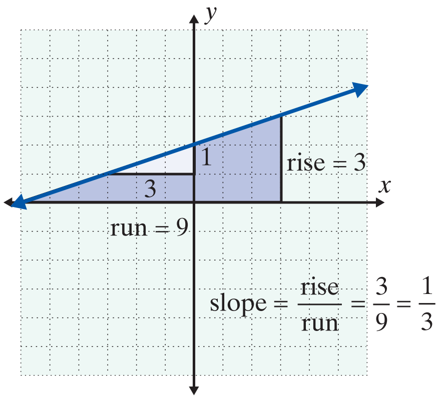
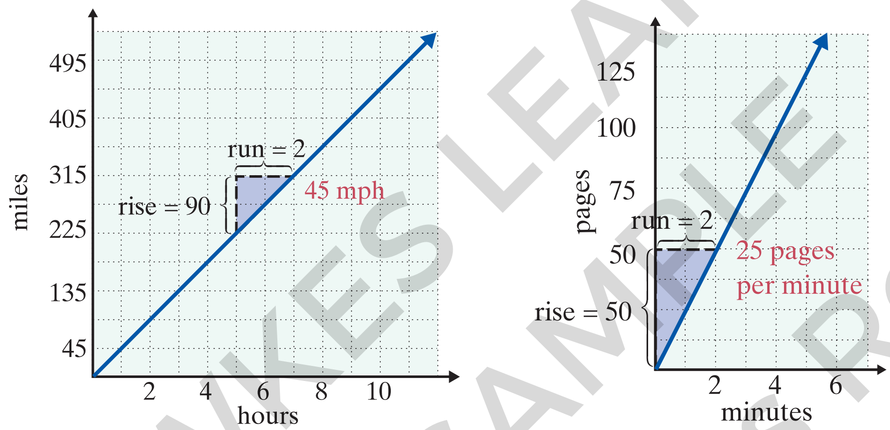


Figure 3

The concept of slope also relates to situations that involve **rate of change**. For example, the graphs in Figure 4 illustrate slope as miles per hour that a car travels and as pages per minute that a printer prints.



slope = rate of change

$$= \frac{90 \text{ miles}}{2 \text{ hours}}$$

$$= 45 \text{ mph}$$

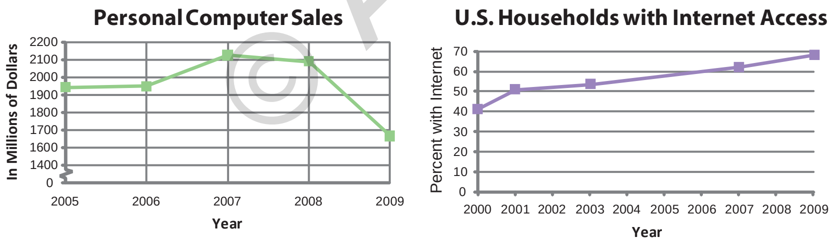
slope = rate of change

$$= \frac{50 \text{ pages}}{2 \text{ minutes}}$$

$$= 25 \text{ pages per minute}$$

Figure 4

In general, the ratio of a change in one variable (say y) to a change in another variable (say x) is called the **rate of change of y with respect to x**. Figure 5 shows how the rate of change (the slope) can change over periods of time.



Source: Consumer Electronics Association

Source: U.S. Dept. of Commerce

Figure 5

Objective B

Calculating the Slope

Consider the line $y = 2x + 3$ and two points on the line $P_1(-2, -1)$ and $P_2(2, 7)$ as shown in Figure 6. (**Note:** In the notation P_1, 1 is called a **subscript** and P_1 is read "P sub 1". Similarly, P_2 is read "P sub 2." Subscripts are used in "labeling" and are not used in calculations.)

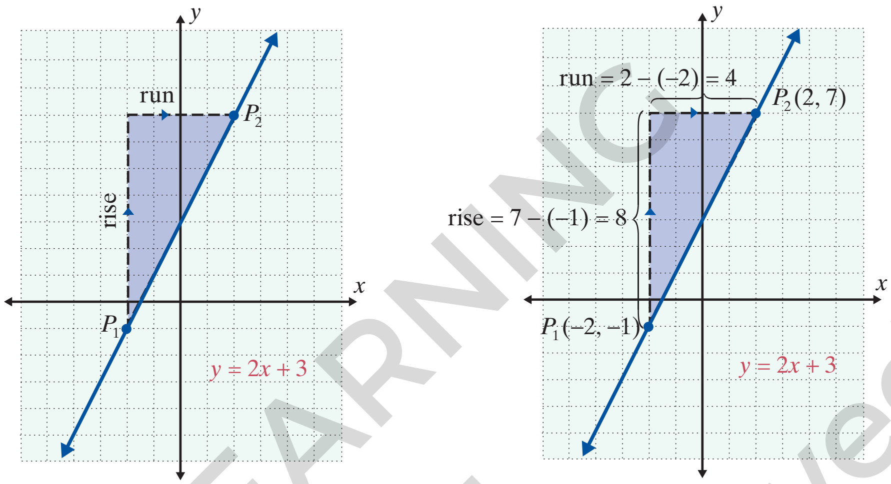


Figure 6

For the line $y = 2x + 3$ and using the points $(-2, -1)$ and $(2, 7)$ that are on the line,

$$\textbf{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}} = \frac{7-(-1)}{2-(-2)} = \frac{8}{4} = 2.$$

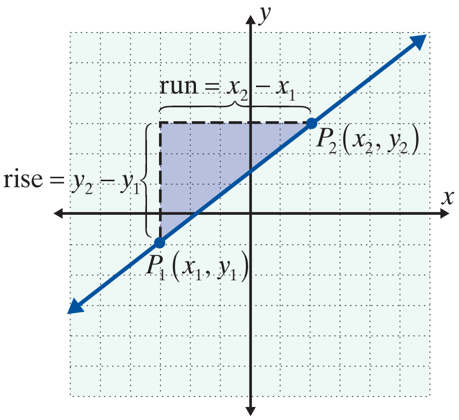
From similar illustrations and the use of subscript notation, we can develop the following formula for the slope of any line.

Slope

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on a line. The **slope** can be calculated as follows.

$$\textbf{slope} = m = \frac{\textbf{rise}}{\textbf{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note: The letter m is standard notation for representing the slope of a line.



$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Figure 7

Example 1

Finding the Slope of a Line

Find the slope of the line that contains the points $(-1, 2)$ and $(3, 5)$, and then graph the line.

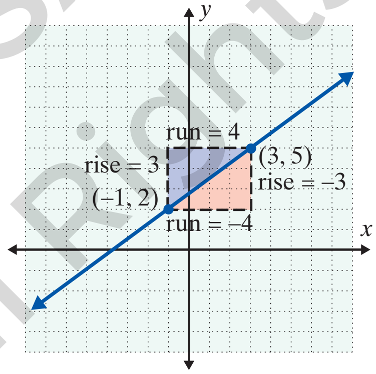
Solution

Using $\underset{(x_1,\, y_1)}{(-1, 2)}$ and $\underset{(x_2,\, y_2)}{(3, 5)}$,

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{3 - (-1)} = \frac{3}{4}$$

Or, using $\underset{(x_1,\, y_1)}{(3, 5)}$ and $\underset{(x_2,\, y_2)}{(-1, 2)}$,

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{-1 - 3} = \frac{-3}{-4} = \frac{3}{4}$$



Now work margin exercise 1.

1. Find the slope of the line that contains the points $(3, -1)$ and $(4, 2)$, and then graph the line.

As we see in Example 1, **the slope is the same even if the order of the points is reversed**. The important part of the procedure is that **the coordinates must be subtracted in the same order in both the numerator and the denominator**.

In general,

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

2. Find the slope of the line that contains the points $(0,5),(4,2)$.

Example 2

Finding the Slope of a Line

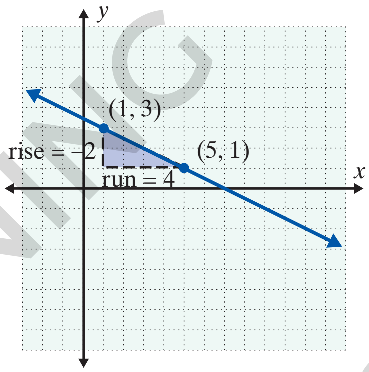
Find the slope of the line that contains the points $(1,3)$ and $(5,1)$, and then graph the line.

Solution

Using $\left(\color{red}{1},\color{red}{3}\right)$ and $\left(\color{blue}{5},\color{blue}{1}\right)$,

$\quad\quad\quad\left(x_1,y_1\right)\quad\quad\left(x_2,y_2\right)$

$$\text{slope} = m = \frac{1-3}{5-1} = \frac{-2}{4} = -\frac{1}{2}$$



Now work margin exercise 2.

notes

- Lines with **positive slope go up** (increase) as we move along the line from left to right.
- Lines with **negative slope go down** (decrease) as we move along the line from left to right.

Objective C Slopes of Horizontal and Vertical Lines

Suppose that two points on a line have the same y-coordinate, such as $(-2,3)$ and $(5,3)$. Then the line through these two points will be **horizontal** as shown in Figure 8. In this case, the y-coordinates of points on the horizontal line are all 3, and the equation of the line is simply $y=3$. The slope is

$$m = \frac{3-3}{5-(-2)} = \frac{0}{7} = 0.$$

For any horizontal line, all of the y-values will be the same. Consequently, the formula for slope will always have 0 in the numerator. Therefore, **the slope of every horizontal line is 0**.

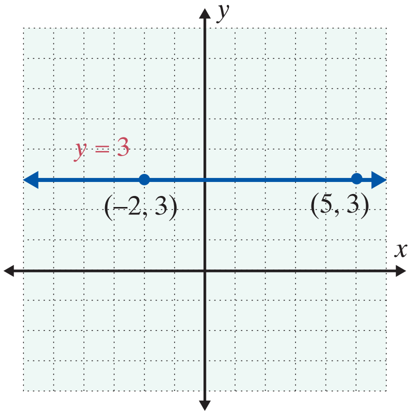


Figure 8

If two points have the same x-coordinates, such as $(1, 3)$ and $(1, -2)$, then the line through these two points will be **vertical** as in Figure 9. The x-coordinates for every point on the vertical line are all 1, and the equation of the line is simply $x = 1$. The slope is

$$m = \frac{-2-3}{1-1} = \frac{-5}{0}, \text{ which is } \textbf{undefined}.$$

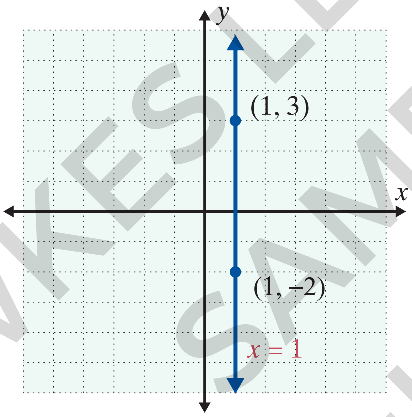


Figure 9

Horizontal and Vertical Lines

The following two general statements are true for horizontal and vertical lines.

1. For **horizontal lines** (of the form $\boldsymbol{y = b}$), the **slope is 0**.
2. For **vertical lines** (of the form $\boldsymbol{x = a}$), the **slope is undefined**.

3. Find the equation and slope of the indicated line.

a. horizonal line through $(3,-2)$

b. vertical line through $(2,4)$

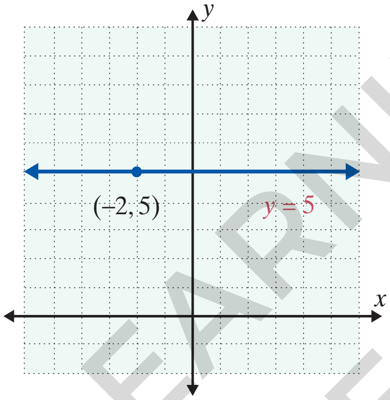
Example 3

Slopes of Horizontal and Vertical Lines

a. Find the equation and slope of the horizontal line through the point $(-2, 5)$.

Solution

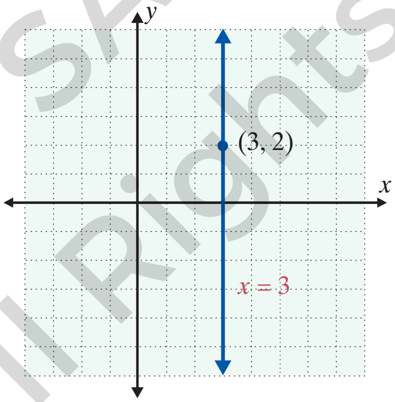
The equation is $y = 5$ and the slope is 0.



b. Find the equation and slope of the vertical line through the point $(3, 2)$.

Solution

The equation is $x = 3$ and the slope is undefined.



Now work margin exercise 3.

Objective D Slope-Intercept Form

There are certain relationships between the coefficients in the equation of a line and the graph of that line. For example, consider the equation

$$y = 5x - 7.$$

First, find two points on the line and calculate the slope. $(0, -7)$ and $(2, 3)$ both satisfy the equation.

$$\text{slope} = m = \frac{3-(-7)}{2-0} = \frac{10}{2} = 5$$

Observe that the slope, $m = 5$, is the same as the coefficient of x in the equation $y = 5x - 7$. This is not just a coincidence. In fact, if a linear equation is solved for y, then the coefficient of x will always be the slope of the line.

For $y = mx + b$, m is the Slope

For an equation in the form $\boldsymbol{y = mx + b}$, the slope of the line is m.

For the line $y = mx + b$, the point where $x = 0$ is the point where the line crosses the y-axis. Recall that this point is called the ***y*-intercept**. By letting $x = 0$, we get

$$y = mx + b$$
$$y = m \cdot 0 + b$$
$$y = b.$$

Thus the point $(0, b)$ is the y-intercept. The concepts of slope and y-intercept lead to the following definition.

Slope-Intercept Form

$\boldsymbol{y = mx + b}$ is called the **slope-intercept form** for the equation of a line, where m is the **slope** and $(0, b)$ is the ***y*-intercept**.

As illustrated in Example 4, an equation in the **standard form**

$$Ax + By = C \text{ with } B \neq 0$$

can be written in the slope-intercept form by solving for y.

Example 4

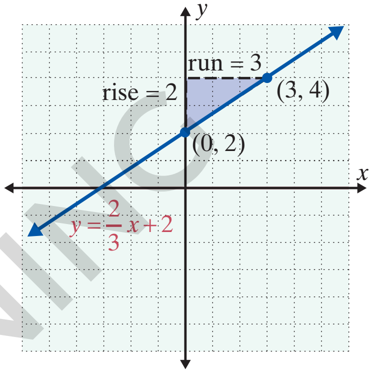
Using the Form $y = mx + b$

a. Find the slope and y-intercept of $-2x + 3y = 6$ and graph the line.

Solution

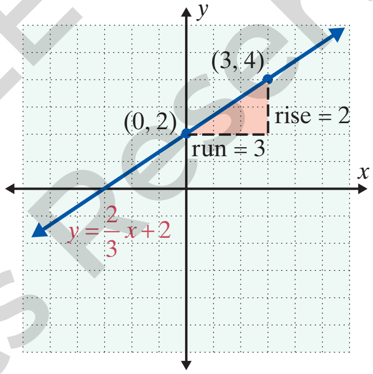
Solve for y.

$$-2x + 3y = 6$$
$$3y = 2x + 6$$
$$\frac{3y}{3} = \frac{2x}{3} + \frac{6}{3}$$
$$y = \frac{2}{3}x + 2$$



Thus $m = \frac{2}{3}$, which is the slope, and b is 2, making the y-intercept equal to $(0, 2)$.

As shown in the graph, if we "rise" 2 units up and "run" 3 units to the right **from the** ***y*-intercept** $(0, 2)$, we locate another point $(3, 4)$. The line can be drawn through these two points. **Note:** As shown in the graph on the right, we could also first "run" 3 units right and "rise" 2 units up from the y-intercept to locate the point $(3, 4)$ on the graph.

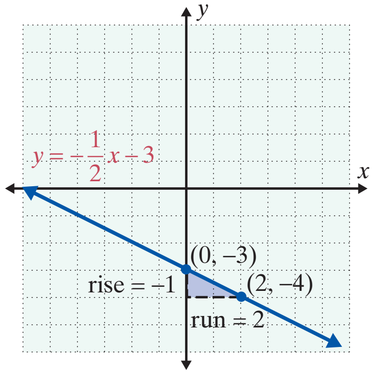


b. Find the slope and y-intercept of $x + 2y = -6$ and graph the line.

Solution

Solve for y.

$$x + 2y = -6$$
$$2y = -x - 6$$
$$\frac{2y}{2} = \frac{-x}{2} - \frac{6}{2}$$
$$y = -\frac{1}{2}x - 3$$



Thus $m = -\frac{1}{2}$, which is the slope, and b is -3, making the y-intercept equal to $(0, -3)$.

We can treat $m=-\frac{1}{2}$ as $m=\frac{-1}{2}$ and the "rise" as -1 and the "run" as 2. Moving from $(0,-3)$ as shown in the graph above, we locate another point $(2,-4)$ on the graph and draw the line.

c. Find the equation of the line through the point $(0,-2)$ with slope $\frac{1}{2}$.

Solution

Because the x-coordinate is 0, we know that the point $(0,-2)$ is the y-intercept. So $b=-2$. The slope is $\frac{1}{2}$. So $m=\frac{1}{2}$. Substituting in slope-intercept form $y=mx+b$ gives the result: $y=\frac{1}{2}x-2$.

Now work margin exercise 4.

4. a. Find the slope and y-intercept of $-4x+2y=12$ and graph the line.

b. Find the slope and y-intercept of $3x+2y=-10$ and graph the line.

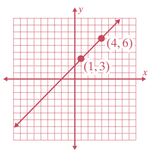
c. Find the equation of the line through the point $(0,-3)$ with a slope of $\frac{2}{3}$.

Practice Problems

1. Find the slope of the line through the two points $(1, 3)$ and $(4, 6)$. Graph the line.
2. Find the equation of the line through the point $(0, 5)$ with slope $-\frac{1}{3}$.
3. Find the slope and y-intercept for the line $2x+y=7$.
4. Write the equation for the horizontal line through the point $(-1, 3)$. What is the slope of this line?
5. Write the equation for the vertical line through the point $(-1, 3)$. What is the slope of this line?

Practice Problem Answers

1. $m=1$



2. $y=-\frac{1}{3}x+5$

3. $m=-2$; y-intercept $=(0, 7)$

4. $y=3$; slope is 0

5. $x=-1$; slope is undefined

Exercises 6.3

Find the slope of the line determined by each set of points. See Examples 1 and 2.

1. $(2, 4); (1, -1)$

2. $(1, -2); (1, 4)$

3. $(-6, 3); (1, 2)$

4. $(-3, 7); (4, -1)$

5. $(-5, 8); (3, 8)$

6. $(-2, 3); (-2, -1)$

7. $(5, 1); (3, 0)$

8. $(0, 0); (-2, -3)$

9. $\left(\frac{3}{4}, \frac{3}{2}\right); (1, 2)$

10. $\left(4, \frac{1}{2}\right); (-1, 2)$

11. $\left(\frac{3}{2}, \frac{4}{5}\right); \left(-2, \frac{1}{10}\right)$

12. $\left(\frac{7}{2}, \frac{3}{4}\right); \left(\frac{1}{2}, -3\right)$

Determine whether each equation represents a horizontal line or vertical line and give its slope. Graph the line. See Example 3.

13. $y = 5$

14. $y = -2$

15. $x = -3$

16. $x = 1.7$

17. $3y = -18$

18. $4x = 2.4$

19. $-3x + 21 = 0$

20. $2y + 5 = 0$

Write each equation in slope-intercept form. Find the slope and the *y*-intercept, and then use them to draw the graph. See Example 4.

21. $y = 2x - 1$

22. $y = 3x - 4$

23. $y = 5 - 4x$

24. $y = 4 - x$

25. $y = \frac{2}{3}x - 3$

26. $y = \frac{2}{5}x + 2$

27. $x + y = 5$

28. $x - 2y = 6$

29. $x + 5y = 10$

30. $4x + y = 0$

31. $4x + y + 3 = 0$

32. $2x + 7y + 7 = 0$

33. $2y - 8 = 0$

34. $3y - 9 = 0$

35. $2x = 3y$

36. $4x = y$

37. $3x + 9 = 0$

38. $4x + 7 = 0$

39. $5x - 6y = 18$

40. $3x + 6 = 6y$

41. $5 - 3x = 4y$

42. $5x = 11 - 2y$

43. $6x + 4y = -8$

44. $7x + 2y = 4$

45. $6y = -6 + 3x$

46. $4x = 3y - 7$

47. $5x - 2y + 5 = 0$

48. $6x + 5y = -15$

In reference to the equation $y = mx + b$, sketch the graph of three lines for each of the two characteristics listed below.

49. $m > 0$ and $b > 0$

50. $m < 0$ and $b > 0$

51. $m > 0$ and $b < 0$

52. $m < 0$ and $b < 0$

Find an equation in slope-intercept form for the line passing through the given point with the given slope. See Example 4.

53. $(0, 3);\ m = -\frac{1}{2}$

54. $(0, 2);\ m = \frac{1}{3}$

55. $(0, -3);\ m = \frac{2}{5}$

56. $(0, -6);\ m = \frac{4}{3}$

57. $(0, -5);\ m = 4$

58. $(0, 9);\ m = -1$

59. $(0, -4);\ m = 1$

60. $(0, 6);\ m = -5$

61. $(0, -3);\ m = -\frac{5}{6}$

62. $(0, -1);\ m = -\frac{3}{2}$

The graph of a line is shown with two points highlighted. Find a. the slope, b. the y-intercept (if there is one), and c. the equation of the line in slope-intercept form.

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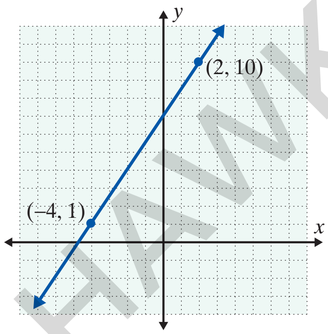

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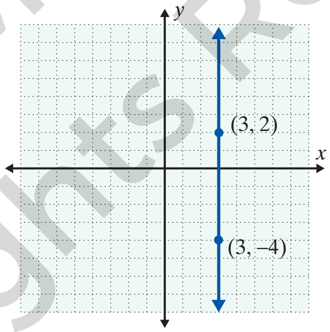

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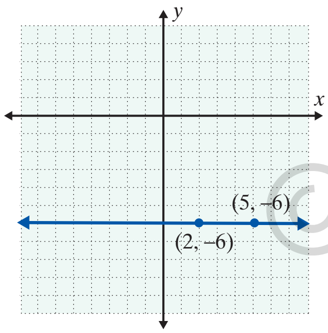

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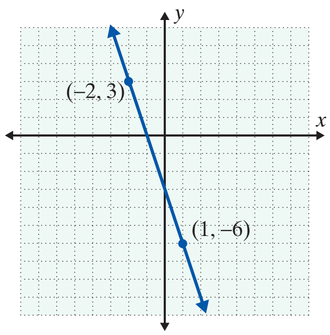

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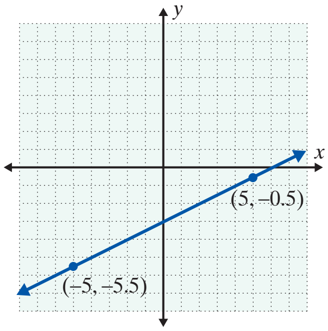

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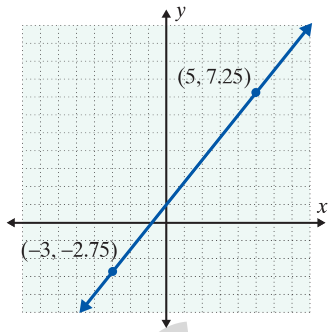

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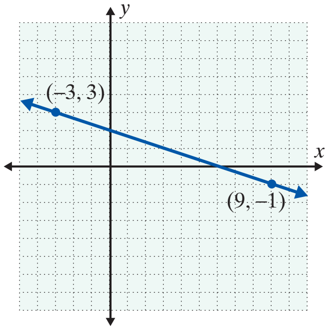

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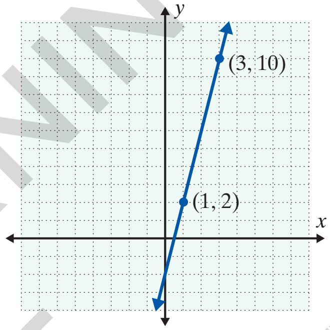


Points are said to be *collinear* if they are on a straight line. If points are collinear, then the slope of the line through any two of them must be the same (because the line is the same line). Use this idea to determine whether or not the three points in each of the sets are collinear.

71. $\{(-1, 3), (0, 1), (5, -9)\}$

72. $\{(-2, -4), (0, 2), (3, 11)\}$

73. $\{(-2, 0), (0, 30), (1.5, 5.25)\}$

74. $\{(-1, -7), (1, 1), (2.5, 7)\}$

75. $\left\{\left(\frac{2}{3}, \frac{1}{2}\right), \left(0, \frac{5}{6}\right), \left(-\frac{3}{4}, \frac{29}{24}\right)\right\}$

76. $\left\{\left(\frac{3}{2}, -\frac{1}{3}\right), \left(0, \frac{1}{6}\right), \left(-\frac{1}{2}, \frac{3}{4}\right)\right\}$

Solve the following word problems.

77. Buying a New Car: John bought his new car for $35,000 in the year 2007. He knows that the value of his car has depreciated linearly. If the value of the car in 2010 was $23,000, what was the annual rate of depreciation of his car? Show this information on a graph. (When graphing, use years as the *x*-coordinates and the corresponding values of the car as the *y*-coordinates.)

78. Cell Phone Usage: The number of people in the United States with mobile cellular phones was about 180 million in 2004 and about 286 million in 2009. If the growth in mobile cellular phones was linear, what was the approximate rate of growth per year from 2004 to 2009. Show this information on a graph. (When graphing, use years as the *x*-coordinates and the corresponding number of users as the *y*-coordinates.)

79. Internet Usage: The given table shows the estimated number of internet users from 2004 to 2008. The number of users for each year is shown in millions.

a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

Source: International Telecommunications Union Yearbook of Statistics

Year	Internet Users (in millions)
2004	185
2005	198
2006	210
2007	220
2008	231

80. Urban Growth: The following table shows the urban growth from 1850 to 2000 in New York, NY.

a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

Source: U.S. Census Bureau

Year	Population
1850	515,547
1900	3,437,202
1950	7,891,957
2000	8,008,278

81. Military: The following graph shows the number of female active duty military personnel over a span from 1945 to 2009. The number of women listed includes both officers and enlisted personnel from the Army, the Navy, the Marine Corps, and the Air Force.

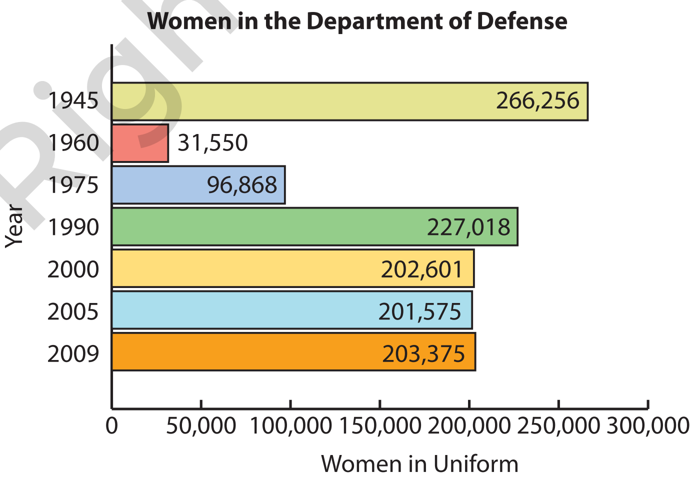
a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

Source: U.S. Dept. of Defense



82. **Marriage:** The following graph shows the rates of marriage per 1000 people in the U.S., over a span from 1920 to 2008.

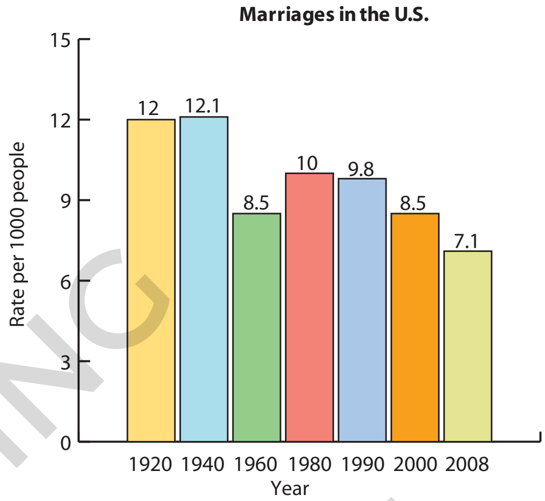
a. Plot these points on a graph.

b. Connect the points with line segments.

c. Find the slope of each line segment.

d. Interpret each slope as a rate of change.

Source : U.S. National Center for Health Statistics



Collaborative Learning

83. The class should be divided into teams of 2 or 3 students. Each team will need access to a digital camera, a printer, and a ruler.

a. Take pictures of 8 things with a defined slope. (**Suggestions:** A roof, a stair railing, a beach umbrella, a crooked tree, etc. Be creative!)

b. Print each picture.

c. Use a ruler to draw a coordinate system on top of each picture. You will probably want to use increments of in. or cm, depending on the size of your picture.

d. Identify the line in each picture whose slope you are calculating and then use the coordinate systems you created to identify the coordinates of two points on each line.

e. Use the points you just found to calculate the slope of the line in each picture.

f. Share your findings with the class.

Writing & Thinking

84. a. Explain in your own words why the slope of a horizontal line must be 0.

b. Explain in your own words why the slope of a vertical line must be undefined.

85. a. Describe the graph of the line $y = 0$.

b. Describe the graph of the line $x = 0$.

86. In the formula $y = mx + b$ explain the meaning of m and the meaning of b.

87. The slope of a road is called a **grade**. A steep grade is cause for truck drivers to have slow speed limits in mountains. What do you think that a "grade of 12%" means? Draw a picture of a right triangle that would indicate a grade of 12%.